J. Fluid Mech. (1982), *vol.* 124, *pp.* 411–437
Printed in Great Britain

A comparative assessment of spectral closures as applied to passive scalar diffusion

By J. R. HERRING, D. SCHERTZER,† M. LESIEUR,‡
G. R. NEWMAN,§ J. P. CHOLLET‡ AND M. LARCHEVEQUE‖
National Center for Atmospheric Research, Boulder, Colorado 80307

(Received 24 July 1981 and in revised form 20 April 1982)

We compare – both analytically and numerically – two related spectral (≡ two-point) closures for the problem of the decay of temperature fluctuations convected by isotropic turbulence. The methods are the test-field model (TFM) (Kraichnan 1971; Newman & Herring 1979) and the eddy-damped quasinormal Markovian (ENQNM) approximation (Orszag 1974; Lesieur & Schertzer 1978). We show that EDQNM may be regarded as a rational approximation to, and simplification of, the TFM, except at small wavenumbers, where an additional eddy-dissipative term is needed to produce satisfactory results for the former. We consider three available methods for determining the relaxation timescales: (i) comparison with experiments, (ii) comparison with the direct-interaction approximation (DIA) in thermal equilibrium, and (iii) comparison with DIA at very small wavenumber, where it is believed to represent the dynamics properly. Comparison with both large Reynolds number and wind-tunnel Reynolds numbers is presented. For the latter, we discuss the relationship of the present theoretical results to the experiments of Warhaft & Lumley (1978) and Sreenivasan *et al.* (1980), and to the theoretical analysis of Corrsin (1964), Kerr & Nelkin (1980) and Antonopolos-Domis (1981).

1. Introduction

This paper examines the logical basis and computability of two related spectral closures, as applied to the study of the decay of isotropic turbulence convecting a passive scalar. The procedures studied are the test-field model (TFM) (Kraichnan 1971; Newman & Herring 1979; Larcheveque *et al.* 1980) and the eddy-damped quasinormal Markovian (EDQNM) approximation (Orszag 1970). This *genre* of closures concerns itself with the dynamics of interacting scales of motion, and hence may serve as a logical starting point for understanding flows that suffer rapid changes, as well as inertial- and dissipation-range phenomena. They may be shown to be fully realizable in the sense of satisfying all relevant Schwartz inequalities for covariances, as well as positivity of energy spectra. Their straightforward application to interesting flows requires the solution of an integro-differential equation, whose complexity at first sight appears just as formidable as the full equations of motion; however, the theoretical equations for the energy spectra are much smoother (in space and time) than the latter, and this smoothness may be exploited to reduce greatly the

† Permanent address: Direction de la Météorologie, Paris.
‡ Permanent address: Institut de Mécanique de Grenoble, Université de Grenoble.
§ Permanent address: AVCO Systems Division, 201 Lowell Street, Wilmington, Massachusetts 01887.
‖ Permanent address: Laboratoire de Météorologie Dynamique, Paris.

computational labour needed for solutions. Algorithms for spectral closures are thus (at wind-tunnel Reynolds numbers $R_\lambda \approx 40$) a factor of $> 10^3$ faster than full spectrum simulations; at larger R_λ they are even more competitive. Thus, such methods are intermediate in computability between second-order modelling and full large-eddy spectral simulation.

The spectral closures may be profitably used to obtain insight into the degree of universality of certain parameters of second-order modelling (such as the Rotta return to isotropy (Herring 1974; Schumann & Herring 1976; Cambon, Jeandel & Mathieu 1980)). Perhaps their earliest use was to deduce the behaviour of energy dissipation as a function of total energy and integral lengthscale (Rotta 1951). On the other hand, their direct application to real, inhomogeneous flows becomes quite complicated. Second-order modelling is then more attractive (if at all justified). Spectral closures (transformed into configuration space) may still be useful in gaining insight into such things as the structure of the turbulent-diffusion coefficient (Herring 1973). These coefficients appear as integrals of the spectra, weighted by turbulent timescales. Both of the latter are ingredients of the one-point closures (for a more complete discussion see Leslie 1973, chap. 15).

Similarly, in employing large-eddy simulation (subgrid-scale modeling) methods, the closures may be used to deduce proper forms of sub-grid-scale eddy parameters. This point has been recently discussed extensively by Leslie & Quarini (1978), Basdevant, Lesieur & Sadourny (1978), and Chollet & Lesieur (1981).

Before recording the closure equations, it is worth recalling certain general physical requirements that useful spectral closures should possess beyond energy positivity noted earlier and Kolmogorov scaling at inertial-range wavenumbers. Given the instability of any laminar state, it is plausible to expect of such methods a tendency to redistribute the scales of motion in both magnitude and direction so that all degrees of freedom of the fluid (all scale sizes) are equally probable, consistent with the constraints of energy conservation. This concept can be made precise only for inviscid flows with truncated equations, in which case the eventual statistical state is one of energy equipartition (Lee 1952; Kraichnan 1958; Rose & Sulem 1978). The kinetic-energy spectrum $E(k, t)$ is then given by

$$E(k,t) = Ck^2, \tag{1.1}$$

where k is the wavenumber. Another important expected feature is that small scales act on large scales primarily by draining their energy *via* an eddy viscosity. At very small wavenumbers, the above ideas of equipartition and eddy viscosity lead us to expect that the decay of the energy spectrum should be given by

$$\dot{E}(k,t) = T(k,t) = A(t)\,k^4 - 2(\nu+\nu_{\text{eddy}})\,k^2\,E(k,t). \tag{1.2}$$

Here $A(t)$ and ν_{eddy} are yet-to-be-determined functionals of energy $E(k, t)$. Also ν_{eddy} (and the eddy diffusivity K_{eddy}) may depend on k as $k \to 0$ through $E(k, t)$; this point will be discussed in more detail in §3.1. Equation (1.2) implies that the effects of small scales on the large cannot be totally viscous, but must input energy via a 'beating' interaction (the first term in (1.2)) leading primarily to an input into large scale. As noted by Lesieur & Schertzer (1978), (1.2), together with the existence of an inertial range at large k, essentially determines the decay law of total energy. The same arguments apply to the passive scalar. $T(k, t)$, as given by (1.2), depends on $E(k, t)$ through $A(t)$ non-locally (i.e. as a functional integral). This feature, essential for predicting total energy decay, is not obtained by other more phenomenological closures such as those of Heisenberg (1948), Oboukhov (1941), Leith (1968), Howells

(1960) or Bell & Nelkin (1977). The reason lies in the fact that these models have strictly wavenumber-local interactions so that $T(k, t) \to k^{\frac{3}{2}}[E(k, t)]^{\frac{3}{2}}$ as $k \to 0$. To our knowledge, only the closures of the present *genre* possess the non-local property and hence are capable of explaining adequately the laws of free turbulent decay.

Section 2 records the equations to be studied and discusses their relation to other more fundamental closures – the direct-interaction approximation (DIA) (Kraichnan 1959) and the Lagrangian-history direct-interaction approximation (LHDIA) (Kraichnan 1965). It turns out that the present methods – especially as applied to the scalar problem – are free neither from arbitrary constants nor from arbitrary assumptions about structural forms of the triple-moment correlation times. Some of this arbitrariness may be eliminated by an appeal to the more fundamental theories such as the DIA or LHDIA. However, which of the latter is a more secure starting point is not clear at present. For this reason, we carry through numerical calculations for methods 'based' on both DIA and LHDIA. In practice, differences are small.

Some general results of the TFM and EDQNM are described in §3, including a discussion of the issue of non-localness, the dependence of scalar spectra on Prandtl number, the evaluation of the relevant empirical constants, and the relation of the TFM to the EDQNM. We discuss in particular to what extent the EDQNM is an approximation to the TFM. Section 4 compares theory with experiment. We include a comparison with large-Reynolds-number data as well as a moderate- and small-R_λ comparison of the recent scalar-decay experiments of Warhaft & Lumley (1978) and Sreenivasan *et al.* (1980).

2. Theory

2.1. *The eddy-damped quasinormal Markovian* (*EDQNM*) *model*

We consider homogeneous and isotropic decaying turbulence convecting a passive scalar. The kinetic-energy spectrum $E(k, t)$ and temperature-variance spectrum $F(k, t)$ are defined as is customary, so that

$$\tfrac{1}{2}\langle \mathbf{v}^2 \rangle = \int_0^\infty dk\, E(k,t), \tag{2.1}$$

$$\langle \theta'^2 \rangle = \int_0^\infty dk\, F(k,t), \tag{2.2}$$

where $\mathbf{v}$ and θ' are fluctuations from the ensemble mean in the velocity vector and scalar field respectively. $E(k, t)$ and $F(k, t)$ are related to the one-dimensional spectra by

$$\phi_v(k_1) = \frac{1}{2}\int_{k_1}^\infty \frac{dp}{p}\left(1-\left(\frac{k_1}{p}\right)^2\right)E(p,t), \tag{2.3}$$

$$\phi_\theta(k_1) = \frac{1}{2}\int_{k_1}^\infty \frac{dp}{p}\, F(p,t), \tag{2.4}$$

so that

$$\langle \mathbf{v}^2 \rangle = 6\int_0^\infty dp\, \phi_v(p) \quad \text{and} \quad \langle \theta'^2 \rangle = 2\int_0^\infty dp\, \phi_\theta(p).$$

It is perhaps easiest to explain the closures by stating first the quasinormal approximation and then the changes needed to produce the TFM and EDQNM approximation, together with the physical interpretation and ameliorating effects of these changes. The QN approximation for homogeneous isotropic turbulence of energy spectral density $E(k, t)$ convecting a passive scalar field of spectral variance $F(k, t)$ may be written as (Monin & Yaglom, 1975, pp. 275, 287)

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E(k,t) = \int_\Delta \frac{dp\,dq}{q}(xy+z^3)\,E(q,t)\,[k^2E(p,t)\,D^v_{kpq}-p^2E(k,t)\,D^v_{pkq}], \tag{2.5}$$

$$\left(\frac{\partial}{\partial t}+2Kk^2\right)F(k,t) = \int_\Delta \frac{k\,dp\,dq}{q^3}(1-z^2)\,E(q,t)\,[k^2F(p,t)\,D^\theta_{kpq}-p^2F(k,t)\,D^\theta_{pkq}], \tag{2.6}$$

where the triad relaxation times D^v_{kpq} and D^θ_{kpq} are

$$D^v_{kpq} = \int_0^t dt'g^0_v(k,t,t')\frac{U(p,t,t')U(q,t,t')}{U(p,t,t)\,U(q,t,t)}, \quad t' \leqslant t \tag{2.7}$$

$$D^\theta_{kpq} = \int_0^t dt'g^0_\theta(k,t,t')\frac{\Theta(p,t,t')\,U(q,t,t')}{\Theta(p,t,t)\,U(q,t,t)}, \tag{2.8}$$

$$\delta(\mathbf{k}+\mathbf{p})\,U(k,t,t') \equiv \langle \mathbf{u}(\mathbf{k},t)\,.\,\mathbf{u}(\mathbf{p},t')\rangle = (2\pi k^2)^{-1}E(k,t) \quad \text{if} \quad t=t'.$$

$$\delta(\mathbf{k}+\mathbf{p})\,\Theta(k,t,t') \equiv \langle \theta(\mathbf{k},t)\,\theta(\mathbf{p},t')\rangle = (4\pi k^2)^{-1}F(k,t) \quad \text{if} \quad t=t'.$$

In (2.5) and (2.6) the wavenumber integral $\int dp\,dq$ is over all (p, q) for which (k, p, q) can form a triangle, and (x, y, z) are cosines of the interior angles opposite (k, p, q). Also, $\mathbf{u}(\mathbf{k}, t)$ is the spectral transform of $\mathbf{u}(\mathbf{x}, t)$, and $\delta(\mathbf{x})$ is the three-dimensional Dirac δ-function. The functions $g^0_{v,\,\theta}(k, t, t')$ satisfy

$$\left(\frac{d}{dt}+\nu k^2\right)g^0_v(k,t,t') = 0, \tag{2.9}$$

$$\left(\frac{d}{dt}+Kk^2\right)g^0_\theta(k,t,t') = 0, \quad t' \leqslant t \tag{2.10}$$

$$g^0_{v,\,\theta}(k,t,t) = 1.$$

The g^0 are Green functions for pure viscous (or conductive) decay. The superscript is introduced for subsequent convenience. We shall soon introduce the unsuperscripted g to refer to the infinitesimal response function. In the QN approximation, $U(k, t, t')$ and $\Theta(k, t, t')$ are

$$U(k,t,t') = g^0_v(k,t,t')\,U(k,t',t'), \tag{2.11}$$

$$\Theta(k,t,t') = g^0_\theta(k,t,t')\,\Theta(k,t',t'). \tag{2.12}$$

We have written these equations in a form convenient for comparison with the equivalent DIA or abridged Lagrangian-history DIA (ALH) equations (see e.g. Kraichnan 1965). Thus to obtain the DIA we need only to replace (2.9) and (2.10) with more general expressions and introduce equations of motion for $U(k, t, t')$ and $\Theta(k, t, t')$ in place of (2.11) and (2.12). We shall not record them here; they may be found in Newman & Herring (1979). We note here only the DIA structure for the relaxation times D^v_{kpq} and D^θ_{kpq}. The D^θ_{kpq} for the ALH has set to unity all relaxation factors referencing k and p, a condition traceable to the fact that θ is constant along Lagrangian trajectories. Of course, for ALH the ds integrals in (2.7) and (2.8) are Lagrangian historical integrals, in contrast to the Eulerian ones of (2.7) and (2.8).

Let us return to our discussion of the QN approximation. We recall that some time ago Ogura (1962) and O'Brien & Francis (1962) showed that even for rather small Reynolds number R_λ the initial-value problems based on QN failed to maintain positive-spectra $E(k, t)$ and $F(k, t)$. Later Orszag (1970) pointed out that only a small change was needed to assure again positivity of spectra: change $U(k, t', t')$ and

$\Theta(k,t',t')$ in (2.11) and (2.12) to $U(k,t,t)$ and $\Theta(k,t,t)$ respectively. The resulting equations, called the quasinormal Markovian (QNM) equations by Orszag because they may be shown to result from a Markovian model of the nonlinearity. They may serve as a zeroth-order systematic perturbation procedure in the sense of Phythian (1969) and Kraichnan (1971). Tatsumi, Kida & Mizushima (1978) derived this procedure from the multiple-timescale analysis. With the Markovian changes, we may dispense with the historical integrals in (2.7) and (2.8) and write

$$\frac{dD^v_{kpq}}{dt} = 1-(\nu k^2+\nu p^2+\nu q^2)\,D^v_{kpq}, \tag{2.13a}$$

$$\frac{dD^\theta_{kpq}}{dt} = 1-(Kk^2+Kp^2+\nu q^2)\,D^\theta_{kpq}. \tag{2.13b}$$

The QNM theory is still not correct in the inertial range since, as noted by Orszag (1974), its inertial range is k^{-2} instead of $k^{-\frac{5}{3}}$ (for a more complete discussion of this point see also Frisch, Lesieur & Schertzer 1980). Further (as noted by Frisch *et al.* 1980), for smooth initial data the skewness $\langle(\partial u/\partial x)^3\rangle/\langle(\partial u/\partial x)^2\rangle^{\frac{3}{2}}$ becomes singular in a finite time as $\nu\to 0$, an unrealistic dynamical feature. The inertial range error noted above results from taking triple-moments relaxation times (i.e. D^v_{kpq} and D^θ_{kpq}), as determined entirely by molecular dissipation, through (2.9) and (2.10). Roughly speaking, we may correct for this by augmenting molecular dissipation in these equations by an appropriate eddy dissipation. Thus we replace (2.9) and (2.10) by

$$\left(\frac{d}{dt}+\nu k^2+\tilde{\lambda}\,\mu_k\right)g_v(k,t,t') = 0, \tag{2.9$'$}$$

$$\left(\frac{d}{dt}+Kk^2+\tilde{\lambda}'\mu_k\right)g_\theta(k,t,t') = 0. \tag{2.10$'$}$$

Physically, we expect μ_k to measure the effects of strain due to scales larger than k^{-1} on mode $\mathbf{k}$. Thus, a possible choice for μ_k (appropriate at large k) is (Pouquet, Lesieur & André 1975)

$$\mu_k^2 = \int_0^k p^2\,dp\,E(p,t). \tag{2.14}$$

In (2.9′) and (2.10′), $\tilde{\lambda}$ and $\tilde{\lambda}'$ are arbitrary constants whose values must be determined through either an appeal to experiment or a comparison with more secure theoretical concepts. At this stage, we should note that our changes induced by including the μ_k in the g-equations have fundamentally altered the interpretation of $U(k,t,t')$ and $\Theta(k,t,t')$: they are now insensitive to large-scale translation, contrary to Eulerian non-simultaneous time covariances. For the latter, a new equation must be introduced; an analysis via a Langevin model (Kraichnan 1971) suggests

$$\left(\frac{d}{dt}+\nu k^2\right)U_{\mathrm{E}}(k,t,t') = -\pi\int_\Delta \frac{dp\,dq}{q}(xy+z^3)\,E(q,t)\,D^v_{pkq}\,U_{\mathrm{E}}(k,t,t'),$$

where the subscript E denotes Eulerian.

Our equations to determine $E(k,t)$ and $F(k,t)$ are now (2.5) and (2.6) with the Ds determined by

$$\frac{dD^v_{kpq}}{dt} = 1-[\nu(k^2+p^2+q^2)+\tilde{\lambda}(\mu_k+\mu_p+\mu_q)]\,D^v_{kpq}, \tag{2.7$'$}$$

$$\frac{dD^\theta_{kpq}}{dt} = 1-[K(k^2+p^2)+\nu q^2+\tilde{\lambda}'(\mu_k+\mu_p)+\tilde{\lambda}''\mu_q]\,D^\theta_{kpq}, \tag{2.8$'$}$$

$$\tilde{\lambda}'' = \tilde{\lambda}.$$

We have eliminated the gs after using (2.9′), (2.10′) and the Markovian versions of (2.11) and (2.12). The parameter $\tilde{\lambda}''$ (at present $\equiv \tilde{\lambda}$) has been introduced into (2.8′) for future development.

The above set is called the eddy-damped quasinormal Markovian (EDQNM) approximation by Orszag (1974). It possesses the pleasing properties of spectral positivity, Kolmogorov inertial ranges, energy equipartition for inviscid flows and computational ease. We shall soon show that they also satisfy (1.2). Note, however, that the choices (2.7′) and (2.8′) are by no means unique with respect to the above virtues: any positive set [D^{v}_{kpq} and D^{θ}_{kpq}, symmetric in (p, k)] will yield all these properties except the Kolmogorov range. Given the large-k form of the (2.14), which assures the latter property, this leaves six (instead of two) arbitrary constants, i.e. six quantities like the lambdas. We have been guided in limiting this choice by comparison with perturbation theories and other general principles. For example, we have taken all the lambda multipliers of the inverse timescales (μ_k, μ_p, μ_q) in (2.7′) to be equal to the same value $\tilde{\lambda}$. This is justified only if we assert a kind of fluctuation–dissipation theorem for the 'Lagrangian' $U(k, t, t')$ and $g(k, t, t')$ entering our Markovianized version of (2.7). The same remark applies also to the fact that the same multiplier is used for μ_k and μ_p in (2.8'). Finally, we have taken the same scale factor for μ_q in (2.8′) as in (2.7′); this is suggested by comparing (2.7) with (2.8). We shall shortly suggest that the cogency for this choice is not compelling.

Perhaps the best way to state the case for our present choice for the $\tilde{\lambda}$s is to say that it results from a comparison of EDQNM with DIA for a particular problem for which DIA is thought to be accurate, the problem of relaxation of small departures from absolute equilibrium in the interaction of modes confined to a thin shell, a procedure used by Kraichnan (1971) in his derivation of the TFM. This procedure could also be used to determine the values of $\tilde{\lambda}$ and $\tilde{\lambda}'$, but we postpone this determination until the next section.

Another more heuristic and perhaps cleaner route to obtain the EDQNM is as follows. First, record the hierarchy of (simultaneous time) cumulant equations for $[\mathbf{u}(\mathbf{k}, t), \theta(\mathbf{k}, t)]$. The resulting equation of motion for the third cumulants contains fourth cumulants whose role is to damp the former, thereby providing a mechanism for them to remain properly bounded. The EDQNM may be obtained by simply replacing the fourth cumulants as they occur in the equation of motion for the third cumulants by an appropriate μ times the third cumulant. The argument is due to Orszag (1974). The μs are unspecified, positive numbers, and arguments similar to those already made must be invoked for their determination. Following this line of reasoning leaves the parameter $\tilde{\lambda}''$ undetermined in (2.8′). As we shall see, this added flexibility may be useful.

2.2. *The test-field model (TFM)*

The TFM has the same spectral-evolution equations (2.5) and (2.6) but a somewhat more elaborate set to determine D^{v}_{kpq} and D^{θ}_{kpq}. We shall discuss this procedure briefly, since it has adequate exposition already (Kraichnan 1971; Herring & Kraichnan 1972; Sulem, Lesieur & Frisch 1975; Newman & Herring 1979). The basic idea is to determine the buildup time for triple correlations (i.e. the D^{v}_{kpq}) as proportional to the time required for a convected (compressible) test field $\mathbf{w}$ to be distorted by straining through the exchange of excitation between its compressive and solenoidal components. It therefore requires two relaxation frequencies, one for the solenoidal component $[\eta^s(k)]$ and one for the compressive component $\eta^c(k)$. The TFM equation for D^v_{kpq} is

$$\left[\frac{d}{dt}+\nu(k^2+p^2+q^2)+\eta^s(k)+\eta^s(p)+\eta^s(q)\right]D^v_{kpq}=1, \tag{2.15}$$

where

$$\eta^s(k)=\tfrac{1}{4}g^2k\int_\Delta\frac{p\,dp\,dq}{q}(1-y^2)(1-z^2)\,D^G_{pkq}\,E(q,t), \tag{2.16}$$

$$\eta^c(k)=\tfrac{1}{2}g^2k\int_\Delta\frac{p\,dp\,dq}{q}(1-y^2)(1-z^2)\,D^G_{kpq}\,E(q,t), \tag{2.17}$$

$$\left[\frac{d}{dt}+\nu(k^2+p^2+q^2)+\eta^c(k)+\eta^s(p)+\eta^s(q)\right]D^G_{kpq}=1. \tag{2.18}$$

The TFM requires two timescales D^v_{kpq} and D^G_{kpq}. It has additionally a more physically motivated historical time history in which the relaxation frequencies (η^s and η^c) evolve according to their own intrinsic timescales D^G_{kpq}, D^G_{pkq}. Kraichnan (1971) proposed that the scalar memory times be taken from $\eta^c(k)$. This is equivalent to taking scalar memory times from the compressive part of a test field $\mathbf{w}$, where $\mathbf{w}$ is advected by $\mathbf{v}$, and possibly damped by molecular processes. If $\mathbf{w}$ is damped by viscosity, the scalar relaxation rate is η^c. If $\mathbf{w}$ is damped by conductivity – as would seem more logical – then its scalar relaxation rate $\tilde{\eta}^c$ should satisfy an equation like (2.15), except that the dissipative process is K instead of ν. We then have for D^θ_{kpq}

$$\left[\frac{d}{dt}+\kappa(k^2+p^2)+\nu q^2+g^2_\theta[\tilde{\eta}^c(k)+\tilde{\eta}^c(p)]+\tilde{g}^2_\theta\,\tilde{\eta}^s(q)\right]D^\theta_{kpq}=1, \tag{2.19}$$

where g^2_θ and $\tilde{g}^2_\theta$ are new scale factors analogous to $\tilde{\lambda}'$ and $\tilde{\lambda}''$ for the EDQNM. Here

$$\tilde{\eta}^s=\tfrac{1}{4}g^2k\int\frac{p\,dp\,dq}{q}(1-y^2)(1-z^2)\,\tilde{D}^G_{pkq}\,E(q,t), \tag{2.16$'$}$$

$$\tilde{\eta}^c=\tfrac{1}{2}g^2k\int\frac{p\,dp\,dq}{q}(1-y^2)(1-z^2)\,\tilde{D}^G_{kpq}\,E(q,t), \tag{2.17$'$}$$

$$\left[\frac{d}{dt}+K(k^2+p^2)+\nu q^2+\tilde{\eta}^c(k)+\tilde{\eta}^s(p)+\eta^s(q)\right]\tilde{D}^G_{kpq}=1. \tag{2.18$'$}$$

In Newman & Herring (1979) the choice for scalar relaxation times was $[\tilde{\eta}^c,\tilde{\eta}^s]=[\eta^c,\eta^s]$ and $g^2_\theta=1$. The present choice seems more logical, especially if the Prandtl number is quite large. Unfortunately, it is also more complicated to implement, although we shall propose accurate abridgements to offset this complication. Equations (2.15)–(2.19), together with (2.5) and (2.6), complete the TFM set for $E(k,t)$ and $F(k,t)$.

2.3. *Formal similarities and differences in the methods*

Before discussing practical numerical distinctions between the EDQNM and TFM, we point out certain formal differences and similarities that are assessable without numerical computation. At long times and at large Reynolds numbers, we may approximately evaluate the Ds by putting to zero $\dot{D}$ in (2.7′), (2.8′), (2.18) and (2.19). For the TFM we are thus led to a pair of integral equations for $\eta^s(k)$ and $\eta^c(k)$ in place of (2.16)–(2.18). For orientation, we may solve this set approximately by noting that the factor D^{G}_{kpq} under the $dp\,dq$ integrals in (2.16) varies less rapidly than the remaining factors and may then be evaluated at a suitable wavenumber $\bar{p}(k, q)$ and removed from the dp integration. In this approximation we find

$$\eta^{s}(k) \approx \tfrac{1}{2}\eta^{c}(k) \approx \tfrac{1}{4}g^{2}\int_{0}^{\infty} dq\,\bar{D}(k,q)\cdot J\left(\frac{q}{k}\right)q^{2}E(q), \tag{2.20}$$

where

$$J(x) = a^{-5}\left\{(a^{2}-1)^{2}\ln\frac{1+a}{|1-a|}-2a+\frac{10a^{3}}{3}\right\}(1+x^{2})^{-1}, \quad a = 2x(1+x^{2})^{-1},$$

$$\bar{D}(k,q) = [\eta^{s}(k)+\eta^{s}(q)+2\eta^{s}(\bar{p})+\nu(k^{2}+p^{2}+q^{2})]^{-1}.$$

At large k the appropriate value for $\bar{D}$ is $[3\eta^{s}(k)+2\nu k^{2}]^{-1}$. We may approximate the integral in (2.14) by noting that $J(x)$ is relatively flat for $x \leqq 1$, after which it decreases rapidly. This suggests an approximation based on taking $J(x)$ to be a step of height $J(0) = \frac{16}{15}$, $x \leqq 1$. Bearing in mind that $E(k) \sim k^{-\frac{5}{3}}$ is the anticipated approximate application then yields

$$\eta^{s}(k)\,[\eta^{s}(k)+\tfrac{2}{3}\nu k^{2}] = \tfrac{4}{45}\Gamma g^{2}\int_{0}^{k} dq\,q^{2}E(q)\ldots, \tag{2.21}$$

where

$$\Gamma \approx \tfrac{16}{15}\int_{0}^{\infty} J(x)\,x^{\frac{1}{3}}\,dx \approx 1{\cdot}7793.$$

Thus at large t and inertial range k we have rough equivalence between the TFM and EDQNM, provided that we take

$$\tilde{\lambda} \approx 0{\cdot}3977g.$$

From (2.21), for k in the dissipation range, $\eta^{s}(k) \rightarrow 0$, while $\mu(k)$ according to (2.14) does not. At small k, $\bar{D} \rightarrow (\eta^{s}(q)+\eta^{c}(q))^{-1}$ and (2.20) reduces to

$$k^{-2}\eta^{s}(k) = \tfrac{4}{45}g^{2}\int_{k}^{\infty} dq\,E(q)/\eta^{s}(q). \tag{2.22}$$

This low-k behaviour of $\eta^{s}(k)$ is quite similar to that of the DIA. If we parametrize the latter's Green functions exponentially (thereby obtaining an extended Edwards' (1964) theory; see Herring & Kraichnan (1972)), we obtain, instead of (2.16),

$$\mu^{\mathrm{DI}}(k) = \tfrac{2}{15}k^{2}\int_{k}^{\infty} dq\,\frac{E_{v}(q)}{\mu^{\mathrm{DI}}(q)}\left(1-\frac{1}{4}\frac{d\ln D^{\mathrm{DI}}(k,q,q)}{d\ln q}\right), \tag{2.23}$$

where $\tilde{\mu}(k)$ here serves the same role for DI as the $\eta^{s}(k)$ for TFM and

$$D^{\mathrm{DI}}(k,p,q) = [\mu^{\mathrm{DI}}(p)+\mu^{\mathrm{DI}}(q)]^{-1}.$$

Either (2.22) or (2.23) must now be solved as an integral equation for $\eta^{s}(k)$ (*or* $\mu(k)$). *If* $E_{v}(k) \rightarrow k^{n}$ as $k \rightarrow 0$ and $n > 1$, $\eta^{s}(k) \rightarrow k^{2}$ as $k \rightarrow 0$, and similarly for $\tilde{\mu}(k)$. Assuming this form in (2.23) to evaluate $d\ln\theta^{\mathrm{DI}}/d\ln q$ then gives

$$\mu^{\mathrm{DI}}(k) = \tfrac{1}{10}k^{2}\int_{k}^{\infty} dq\,\frac{E(q)}{\mu^{\mathrm{DI}}(q)}. \tag{2.24}$$

Thus, in the limit $k \rightarrow 0$ forcing agreement between DIA and TFM implies

$$g_{v}^{2} = \tfrac{9}{8}. \tag{2.25a}$$

A similar calculation for the scalar field yields

$$\tilde{g}_{\theta}^{2} = \sqrt{\tfrac{15}{16}}\,g_{v}^{-1}. \tag{2.25b}$$

These comparisons with the DIA must be viewed with some scepticism because its exponential parameterization is somewhat suspect.

The EDQNM behaviour at small k, as stated by (2.14), does not have the same non-local structure as (2.22). The latter is not only obtained by the DIA, but is also supported by renormalization-group (RNG) calculations (e.g. see Foster, Nelson & Stephen 1977; Fournier & Frisch 1978). Hence it is probably the more plausible of the two at $k \to 0$. This discussion suggests modifying (2.14) to account for the more satisfactory small-**k** behaviour. A plausible approach toward remedying this defect in the EDQNM is suggested by an examination of (2.20). Note first that $J(x) = x^{-2}J(1/x)$, so that if $J(x) = \Sigma J_n x^n (x \leqslant 1)$

$$\eta^s(k) = \sum_0^\infty J_m \left\{ \int_0^k q^2\, dq \left(\frac{q}{k}\right)^n E(q) + k^2 \int_k^\infty \left(\frac{k}{q}\right)^n E(q)\, dq \right\} \bar{D}. \tag{2.20$'$}$$

Recalling that $D \sim \frac{1}{3}[3\eta^s(k)+2\nu k^2]^{-1}$ *for* $k \to \infty$, and $D \to [3\eta^s(q)+2\nu q^2)^{-1}$ for $k \to 0$ then suggests, in connection with (2.20′) above, that

$$\mu(k)\,[\mu(k)+\tfrac{2}{3}\nu k^2] \approx \lambda^2 \int_0^k q^2\, dq\, E(q) + \lambda^2 k^2 \mu(k) \int_k^\infty dq\, \frac{E(q)}{\mu(q)}. \tag{2.14$'$}$$

In using (2.14′), it would seem plausible to evaluate the last integral by approximating

$$\mu(k) \approx ak^2,$$

where

$$a^2 = \lambda^2 \int_0^\infty dq E(q)/q^2,$$

provided $E(k) \to k^n$ $(n > 1)$ as $k \to 0$.

3. Discussion of prediction of models

3.1. *Qualitative prediction of EDQNM and TFM*

This section reviews briefly certain qualitative predictions shared by EDQNM and TFM: (i) the nature of the large-scale (low-wavenumber) transfer and its implication for eddy viscosity and conductivity, and (ii) the prediction of theory at large Reynolds numbers over a range of Prandtl numbers. These issues have been discussed in some detail elsewhere and we simply collect them here for ready access later.

At small k we may develop the right-hand sides of (2.5) and (2.6) (which we denote as T_v and T_θ respectively) in a Taylor series in k/p. There result (Kraichnan 1976; Lesieur & Schertzer 1978)

$$T_v(k) = \tfrac{14}{15}k^4 \int_0^\infty q^{-2}\, dq\, E^2(q)\, D^v_{kqq} - \nu_{\text{eddy}} k^2 E(k) + \ldots, \tag{3.1}$$

$$T_\theta(k) = \tfrac{4}{3}k^4 \int_0^\infty q^{-2}\, dq\, F(q)\, E(p)\, D^v_{kqq} - K_{\text{eddy}} k^2 F_\theta(k) + \ldots, \tag{3.2}$$

where

$$\nu_{\text{eddy}} = \frac{2}{15}\int_0^\infty dq\, E(q)\, D^v_{kqq}\left(1 - \frac{1}{4}\frac{d \ln D^v_{kqq}}{d \ln q}\right), \tag{3.3}$$

$$K_{\text{eddy}} = \frac{2}{3}\int_0^\infty dq\, E(q)\, D^\theta_{kqq}. \tag{3.4}$$

The form (3.3) differs from that derived by Kraichnan (1976) by a partial integration.

We recall that for long times (near stationarity) the EDQNM and TFM equations for the relaxation times D^v and D^θ are

$$\left.\begin{aligned} D^v_{kpq} &= [\eta^s(k)+\eta^s(p)+\eta^s(q)]^{-1}, \\ D^\theta_{kpq} &= \{g^2_\theta[\eta^c(k)+\eta^c(p)]+\eta^s(q)\}^{-1}, \\ \eta^c(k) &\approx 2\eta^s(k) = (2.17); \end{aligned}\right\} \tag{3.5 TFM}$$

$$\left.\begin{aligned} D^v_{kpq} &= [\mu(k)+\mu(p)+\mu(q)]^{-1}\tilde{\lambda}^{-1}, \\ D^\theta_{kpq} &= \{\lambda'[\mu(k)+\mu(p)]+\tilde{\lambda}''\mu(q)\}^{-1}, \\ \mu(k) &= (2.14'). \end{aligned}\right\} \tag{3.5 EDQNM}$$

The viscous–conductive contribution to (3.5) is here omitted for brevity. We evaluate the as-yet arbitrary constants in (3.5) in §3.2. Equations (3.1) and (3.2) satisfy (1.2), which, as observed earlier, is essential for obtaining the proper decay laws for total energy and temperature variance. Further, (3.3) and (3.4) readily give an estimate for a kind of eddy Prandtl number (Larcheveque *et al.* 1980):

$$Pr^e \equiv \frac{\nu_{\text{eddy}}}{K_{\text{eddy}}} = \frac{\lambda'+\lambda''}{5\lambda}\left\{1-\frac{\int_k^\infty dq\, qE(q)\, dD^v/dq}{4\int_k^\infty dq\, E_v(q)\, D^v(q)}\right\}. \tag{3.6}$$

We have written the eddy Prandtl number Pr^e for the EDQNM and have abbreviated D^v_{kqq} by $D^v(q)$. The equivalent prescription for the TFM is omitted for brevity.

The significance of (3.6) for the real-flow context is not at first transparent, since we are dealing here with energy spectra (E and F), and eddy-viscosity concepts are usually introduced for *amplitudes* ($\mathbf{v}$, θ). The conditions under which (3.6) applies are revealed by considering an initial-value problem $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})+\delta\mathbf{u}$, $\theta = \theta_0+\delta\theta$, where the background fields ($\mathbf{v}_0$, θ_0) are concentrated at very small scales compared with ($\delta\mathbf{v}$, $\delta\theta$). Then, if the statistics of ($\mathbf{u}$, θ) are isotropic, the spectra of ($\delta\mathbf{v}$, $\delta\theta$) are dissipated by $2v_{\text{eddy}}k^2$ and $2K_{\text{eddy}}k^2$ respectively.

Lesieur & Chollet (1980) have considered the case in which the background $E(k)$ is zero (for k less than some given wavenumber k_0) and a continuous $k^{-\frac{5}{3}}$ spectrum for $k \geqq k_0$. In this case, (3.6) may be integrated by parts to give

$$Pr^e = \frac{\tilde{\lambda}'+\tilde{\lambda}''}{6\tilde{\lambda}}. \tag{3.7}$$

This estimate would be increased by a more realistic assumption that the spectrum exterior to $k = 0$ is continuous, instead of increasing stepwise at $k \geqq k_0$. For example, taking

$$E(k) = \delta(k)+E'(k),\ E'(k) = k^n\ (k \leqq k_0), \quad = (k_0/k)^{-\frac{5}{3}}k_0{}^n\ (k \geqq k_0),$$

and

$$D^v(k) = k^{-a}\ (k \leqq k_0), \quad = k^{-b}\ (k \geqq k_0),$$

we find that (3.7) should be multiplied by Γ', where

$$\Gamma' = \frac{6\{1+\frac{1}{4}[a+(4+b)(n-a+1)/(\frac{2}{3}+b)]\}}{5\{1+(n-a+1)/(\frac{2}{3}+b)\}}. \tag{3.8}$$

If we further take $a = 2$, $b = \frac{2}{3}$, as seems indicated by (2.20), we find the above factor ranges from $\frac{9}{5}$ (for $n = 1$) to $\frac{7}{5}$ (for $n = \infty$).

At very large Prandtl number, the equation for the scalar spectrum reduces to (Kraichnan 1968; Newman & Herring 1979)

$$\left(\frac{d}{dt}+2Kk^2\right)F(k) = \Lambda\frac{\partial k}{\partial k}\left(k\frac{\partial}{\partial k}-3\right)kF(k), \tag{3.9}$$

where

$$\Lambda = \frac{2}{15}\int_0^\infty \frac{q^2\,dq\,E(q)}{2\tilde{\lambda}'\mu(k)+\tilde{\lambda}''\mu(q)+\nu q^2}. \tag{3.10}$$

Equation (3.9) is derived by assuming that $k_s = (\epsilon/\nu^3)^{\frac{1}{4}} \ll k$. Again, only the EDQNM formula in the form appropriate for $t \to \infty$ is recorded for brevity. The inertial range for (3.9) is

$$F(k) = N(\nu/\epsilon)^{\frac{1}{2}}C_{\rm B}k^{-1}, \tag{3.11}$$

where the Batchelor constant $C_{\rm B}$ is

$$C_{\rm B} = \frac{1}{3\Lambda}\left(\frac{\epsilon}{\nu}\right)^{\frac{1}{2}} \tag{3.12}$$

and N is the rate of molecular dissipation of the scalar variance.

At very small Prandtl number our procedures collapse into the quasinormal results of Batchelor, Howells & Townsend (1959):

$$F(k) = \frac{NE(k)}{3k^3k^4}. \tag{3.13}$$

In this case the $F(k)$ spectrum is entirely diffusive. We remark that for two-dimensional turbulence the analogue to (3.13) is $F(k) \sim k^{-7}$. See Lesieur, Sommeria & Holloway (1981) for further discussion.

3.2. *Determination of empirical constants*

The constants $(g_v, g_\theta, \tilde{g}_\theta)$ for TFM or $(\tilde{\lambda}, \tilde{\lambda}', \tilde{\lambda}'')$ for EDQNM (see (3.5) above) must be fixed through either a comparison of theory to experiment or a comparison of the present phenomenological theory to a method free of arbitrariness. We recall in this connection that the original presentation of the TFM (Kraichnan 1971) identified g_v by requiring that it give results identical to the DIA for perturbations from absolute equilibrium for the problem of interacting modes in a thin spherical shell. The result was

$$g_v = 1{\cdot}064. \tag{3.14}$$

The same procedure applied to the scalar problem gives

$$4g_\theta^2+\tilde{g}_\theta^2 = 3{\cdot}93. \tag{3.15}$$

The method determines only the total relaxation factor D^θ_{kkk} and hence cannot be used to discriminate between g_θ and $\tilde{g}_\theta$. If we construct the TFM analogous to the DIA, then, as noted in §2, $\tilde{g}_\theta = 1$ and $g_\theta^2 = 0{\cdot}7324$, whereas following the LHDI analogue gives $g_\theta = 0$, $\tilde{g}_\theta^2 = 3{\cdot}93$. We will return to a comparison of these results to observations.

We now discuss methods of determining these constants by appealing to large-Reynolds-number experiments. In this connection it should be noted that, if both energy and scalar variance are inertial, differences between TFM and EDQNM are purely a matter of adjusting scaling constants. We therefore discuss only the simpler EDQNM theory and state at the end of our discussion the relation between the

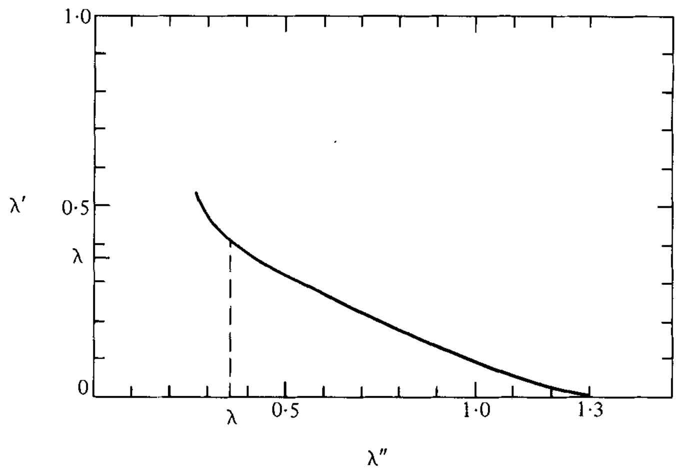


FIGURE 1. EDQNM functional relationship $\tilde{\lambda}' = f(\tilde{\lambda}'')$ necessary for EDQNM to yield a $C_\Theta(\tilde{\lambda}', \tilde{\lambda}'') = 0{\cdot}667$. Here C_Θ is the Oboukhov–Corrsin constant. Underlying calculations according to Quarini (1976).

$(\tilde{\lambda}, \tilde{\lambda}', \tilde{\lambda}'')$ and $(g, G_\theta, \tilde{g}_\theta)$. We shall assume that the kinetic-energy and scalar-variance spectra are given by

$$E(k) = C_K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \quad C_K = 1{\cdot}4, \tag{3.16}$$

$$F(k) = C_\Theta N/\epsilon^{\frac{1}{3}} k^{-\frac{5}{3}}, \quad C_\Theta = 0{\cdot}667, \tag{3.17}$$

respectively. These represent rather well the inertial-range measurements in air of Champagne *et al.* (1977); how well a theory that approaches (3.16), (3.17) asymptotically at $R_\lambda \to \infty$ does at finite R_λ will be discussed in §4. André & Lesieur (1977) have shown that for the EDQNM (3.16) is asymptotically matched by the choice

$$\tilde{\lambda} = 0{\cdot}360, \tag{3.18}$$

which is fairly good agreement with our crude estimate (2.21). For the scalar field we need to determine $(\tilde{\lambda}', \tilde{\lambda}'')$, so a constraint in addition to (3.17) is needed. A possible choice is the Batchelor constant C_B (see (3.11)) for $Pr \to \infty$, but the choice of this constraint may be unwise because of the large scatter in the large-Pr inertial-range observations. Another useful constraint is the eddy Prandtl number Pr^e (see (3.7)) for which experiments (see Fulachier & Dumas 1976) give

$$(0{\cdot}6 \lesssim Pr^e \lesssim 0{\cdot}8). \tag{3.19}$$

The question of interest is, given $C_K = 1{\cdot}4$ and $C_\Theta = 0{\cdot}66$, and to (3·7) for Pr^e, what value of $\tilde{\lambda}'/\tilde{\lambda}''$ gives Pr^e within the experimentally prescribed bounds (3.19)?

The calculations necessary to answer this question have been performed by Quarini (1976). Figure 1 shows the relationship $\tilde{\lambda}' = f(\tilde{\lambda}'')$ implied by $C_\Theta(\tilde{\lambda}', \tilde{\lambda}'') = 0{\cdot}667$. Figure 2 gives $Pr^e(\tilde{\lambda}'/\tilde{\lambda}'')$ (as given by (3.7)) as a function of $\tilde{\lambda}'/\tilde{\lambda}''$, as deduced from Quarini's calculations. The value of Pr^e here subject to the uncertainty noted in the discussion just preceding (3.8). Except for $\tilde{\lambda}'/\tilde{\lambda}'' \to 0$, the value of Pr^e is rather insensitive to $\tilde{\lambda}'/\tilde{\lambda}''$. If Pr^e as given by (3.7) is accurate, it would appear that $\tilde{\lambda}' = 0$ is the best choice. This would fit the LHDI analogy, but would leave to be explained why the advective relaxation time for the scalar $(\tilde{\lambda}'')^{-1} = 0{\cdot}77$ is much less than that for the velocity field $(\tilde{\lambda}^{-1} = 2{\cdot}77)$. Following the LHDI analogy would lead to the conclusion that they were the same.

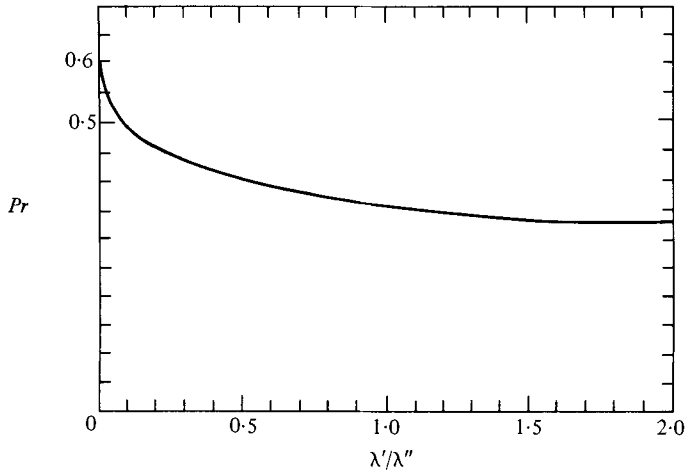


FIGURE 2. Eddy Prandtl number Pr^e as a function of $\tilde{\lambda}'/\tilde{\lambda}''$, where $\tilde{\lambda}'$ and $\tilde{\lambda}''$ are scaling parameters of EDQNM (see (2.8′), (3.5′) and (3.20)). Constraints $C_K = 1{\cdot}4$ and $C_\Theta = 0{\cdot}66$ (i.e. values of Kolmogorov's and the Oboukhov–Corrsin constants) have been imposed. (3.7) is used for Pr^e.

Method	g_v	g_θ^2	$\tilde{g}_\theta^2$
$k \to 0$ comparison with DIA	$\surd\frac{9}{8}$	$\surd\frac{5}{6}$	1
Shell comparison with DIA	1·064	0·7324	1
Comparison with experiment	1·06	0·5 (DIA)	1 (DIA)
		0·0 (ALH)	3·61 (ALH)

TABLE 1

Finally, we state the correspondence between the EDQNM and TFM parameters:

$$2g_\theta^2 = \tilde{\lambda}'/\tilde{\lambda}, \tag{3.20}$$

$$\tilde{g}_\theta^2 = \tilde{\lambda}''/\tilde{\lambda}, \tag{3.21}$$

$$g_v = 2{\cdot}77\tilde{\lambda}. \tag{3.22}$$

Possible assignments of $(g_v, g_\theta^2, \tilde{g}_\theta^2)$ according to various estimates described in this section are summarized in table 1.

4. Some numerical comparisons

4.1. *Large-Reynolds-number results*

We examine first predictions of the theories for large-Reynolds-number and large-Péclet-number free decay. The initial spectra are

$$E(k) = \tfrac{1}{2}F(k) = ak^4(k_0+k)^{-\frac{17}{3}}. \tag{4.1}$$

Here, $\nu = 0{\cdot}001$, $Pr = 0{\cdot}725$ (air), and the computational wavenumber domain is $(0{\cdot}0001, 200)$. The values of (a, k_0) are $(2, 0{\cdot}005)$. Such initial data yield an initial $R_\lambda \approx 2500$, which is typical of planetary boundary turbulence. A set of 40 exponentially spaced knots, in conjunction with B-splines (used collocatively), represented the decay of $E(k, t)$ and $F(k, t)$ on the interpolation interval k_d to an accuracy of better

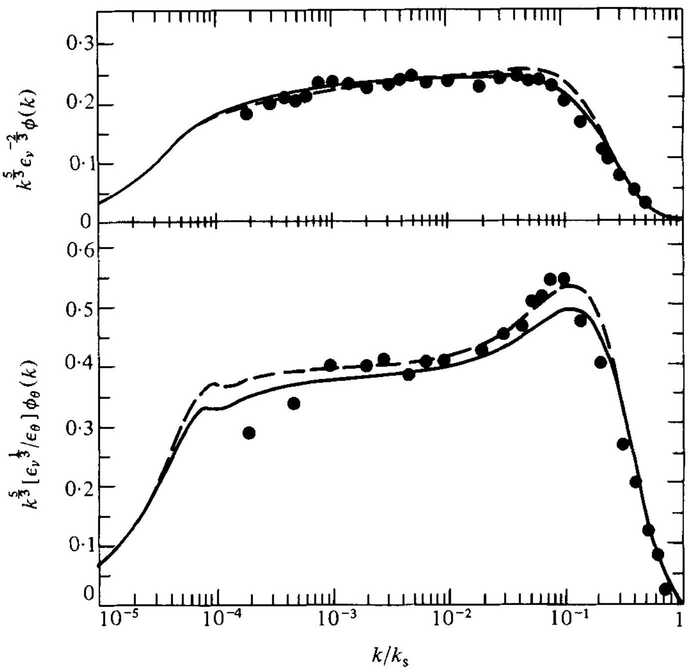


FIGURE 3. Compensated spectra $\alpha = \phi_v(k)\,k^{\frac{5}{3}}\epsilon^{-\frac{2}{3}}$ and $\beta = \phi_\theta(k)\,\epsilon^{\frac{1}{3}}N^{-1}$ (see (2.3) and (2.4)) as functions of k/k_s, $k_s = (\epsilon/\nu^3)^{\frac{1}{4}}$. Points are data of Champagne *et al.* (1977). Two theoretical curves are shown: ——, TFM ($g = 1{\cdot}08$, $g_\theta^2 = 0{\cdot}5$, $g_\theta = 1$); – – –, EDQNM ($\tilde{\lambda} = 0{\cdot}36$, $\tilde{\lambda}' = 0$, $\tilde{\lambda}'' = 3{\cdot}61\tilde{\lambda}$).

than $\sim 1\,\%$, as judged by overall total energy conservation. The large-k range of E and F quickly evolves into approximate self-similar shapes, and we show in figure 3 α' and β' plots (also called compensated spectral plots), which are $\alpha = \phi_v(k)\,k^{\frac{5}{3}}\epsilon^{-\frac{2}{3}}$ and $\beta = \phi_\theta(k)\,k^{\frac{5}{3}}\epsilon^{\frac{1}{3}}N^{-1}$ (see (2.3) and (2.4)) respectively. The points are the data of Champagne *et al.* (1977). The decay time at which the curves are shown ($t \approx 300$ large-eddy circulation time $L\langle u^2\rangle^{-\frac{1}{2}}$, where L is the integral scale) is sufficiently long for the range $k/k_s > 10^{-4}$ to have become completely self-similar. Here $k_s = (\epsilon/\nu^3)^{\frac{1}{4}}$. At smaller k, this is insufficient time for the development of self-similarity; indeed, as noted by Schertzer (1980), for initial conditions (4.1) full self-similarity over all k is not expected. Nonetheless, k_s scaling does appear to render the large-k range (inertial plus dissipation) self-similar. Both ϕ_v and ϕ_θ characteristically exceed the $k^{-\frac{5}{3}}$ law until the beginning of the dissipation range, where a transitional 'bump' appears as **k** enters the dissipation range. The transitional 'bumps' are much more pronounced for the (three-dimensional) energy spectra E and F. This energy (E) bump has already been noticed by André & Lesieur (1977).

The dynamical reason for the spectral bumps may be traced – in the theory – to the special role of the eddy-viscosity (conductivity) terms. The transfer terms of (2.5) and (2.6) may be split into a 'local' part (corresponding to triad interactions such that the ratio of any two wavenumbers of the triad should be larger than a given 'non-localness' parameter δ) and a non-local part corresponding to elongated triads. Among the latter, non-local triads such that $k \ll p \sim q$ give rise to a transfer T_v (or T_θ) obtained from (3.1), (3.2), (3.3), and (3.4) by changing the lower bound 0 of the integrals to k/δ. We may identify an eddy viscosity and conductivity $\nu_e(k, k/\delta)$ and $K_e(k, k/\delta)$ in connection with this lower-bound replacement in (3.3) and (3.4). Then such an eddy viscosity or conductivity begins to decrease abruptly as k exceeds δk_s. The corresponding fluxes

$$\int_k^\infty dk' T_{v,\theta}(k'),$$

constant and positive for $k < \delta k_s$, tend to zero for $k > \delta k_s$. It is reasonable to expect that the local fluxes keep their constant value up to k_s. In the range $(\delta k_s, k_s)$, the $T_v(k)$ transfer, being positive, generates a relative excess of energy, which causes 'erosion' of the lower part of the $k^{-\frac{5}{3}}$ inertial range through the non-local eddy-viscous interactions. We have checked numerically that the energy bump disappears for a calculation that excludes these non-local interactions (δ being taken equal to 0·2). The scalar field may be analysed in the same way: excluding the non-local eddy-diffusive interactions greatly reduces the height of the bump but cannot succeed in making it disappear completely, for any value of δ. Consequently, the scalar bump has its origin in the very non-local interaction discussed above and in others, among which is the viscous-convective k^{-1} proposal of Hill (1978). The latter requires fewer non-local diffusion terms. Comparable bumps in the energy and scalar spectra were observed in the experiment of Mestayer (1980) at $R_\lambda = 600$.

We return to our discussion of the differences between TFM and EDQNM. Consider first the TFM ($g_v = 1{\cdot}06$, $g_\theta^2 = 0{\cdot}5$, $\tilde{g}_\theta^2 = 1$), and EDQNM ($\tilde{\lambda} = 0{\cdot}36$, $\tilde{\lambda}' = 0$, $\tilde{\lambda}'' = 3{\cdot}61$). We examine the velocity-field predictions shown in figure 3(*a*) as the solid and dashed lines respectively. The EDQNM predicts a slightly more enhanced spectral bump for $\phi_v(k)$, with a slightly smaller spectrum in the region $k \approx 10^{-3}k_s$ and a slightly more positive slope leading into the transition region. In terms of a comparison with the data, little real difference appears between the two. The TFM and EDQNM difference that does occur is simply related to their different eddy-damping rates. At very large k, (2.14) gives $\mu(k) \rightarrow (\epsilon/2\nu)^{\frac{1}{2}}$, while $\eta^s(k) \rightarrow 0$, according to (2.16) and (2.17). Recalling that the larger damping rate suppresses energy transfer more, we are led to expect the TFM to transfer energy more efficiently and hence have a smaller effective Kolmogorov constant. We also show $\mu(k)/\eta^s(k)$ for the present run in figure 4. The ordinate scale is at the right.

It is difficult to judge on purely phenomenological grounds which of the above $k \rightarrow \infty$ behaviours is more correct. It seems quite plausible that the limitation on the buildup of triple moments should be proportional to the r.m.s. large-scale strain $(\epsilon/\nu)^{\frac{1}{2}}$. This is not scale-dependent. Such considerations tend to lend credibility to the EDQNM. On the other hand, we would obtain a formula closely similar to the TFM prescription from LHDIA or DIA, whose analytic structure may seem more trustworthy.

The comparison of theory and the data of Champagne *et al.* (1977) for the temperature field is shown in figure 3(*b*). Shown are only those calculations using those TFM and EDQNM coefficients that best match the data ($g_v^2 = 1{\cdot}17$, $g_\theta^2 = 0{\cdot}5$, $\tilde{g}_\theta^2 = 1$) for TFM and ($\tilde{\lambda} = 0{\cdot}36$, $\tilde{\lambda}' = 0$, $\tilde{\lambda}'' = 3{\cdot}61$) for EDQNM. The curves are quite similar, with EDQNM yielding a slightly larger value of the constant β. This difference apparently simply reflects the fact that the overall effective eddy-damping rate is larger for EDQNM than for the TFM ($0{\cdot}36\mu(q)$) as compared with

$$0{\cdot}5[\eta^c(k)+\eta^c(p)+\eta^s(q)],$$

where we recall that $\eta^c(k) \approx 2\eta^s(k)$. Crudely speaking, if we argue that the interactions are effectively local ($k \approx p \approx q$), we would expect the ratio of EDQNM to TFM to be $(3{\cdot}61/3)^{\frac{2}{3}} \approx 1{\cdot}13$, not far different from that indicated in figure 3(*b*).

We have proposed in §3 that the parameters of either theory could be identified

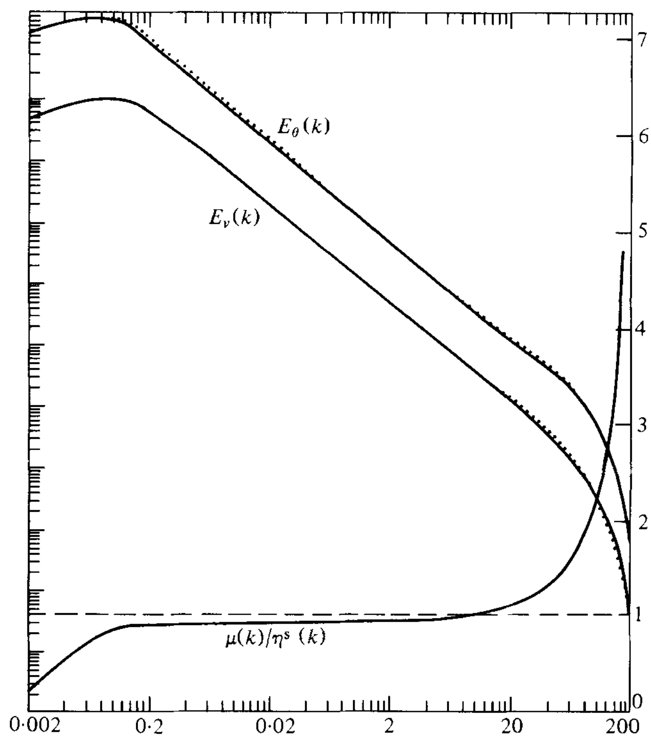


FIGURE 4. Comparison of energy spectra $E(k)$ and $F(k)$ for TFM (——) and EDQNM (· · ·). Parameters are the same as in figure 2. Also shown is ratio of relaxation frequency for the two theories, $\mu(k)/\eta^s(k)$ (ordinate scale to the right). For these runs $k_s = 185·6$.

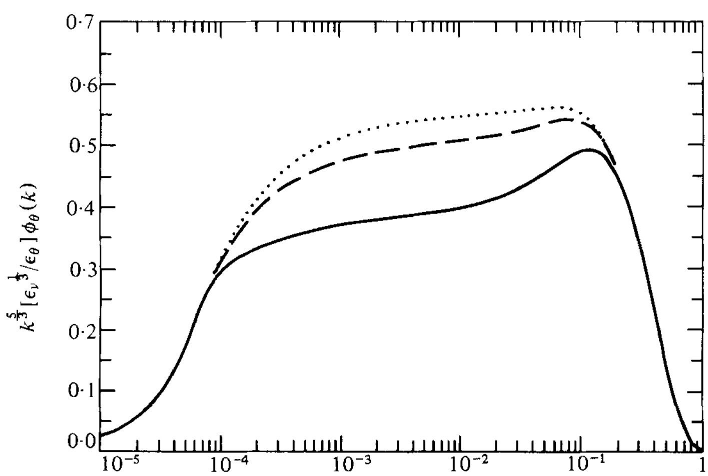


FIGURE 5. Compensated spectra for TFM ($g = 1·08$, $g_\theta^2 = 0·7324$, $\tilde{g}_\theta = 1$) and ($g = 1·07$, $g_\theta^2 = 0·9128$, $g_\theta = 1$), as compared with TFM with g-parameters of figure 3. The first according to 'shell'-absolute equilibrium matching (– – –), the second matches TFM to $k \rightarrow 0$ limit of DIA (. . . . .). See table 1. Initial spectra for this run are $[Ek), F(k)] = 2\pi k^2/(0·01+k)^{-\frac{17}{3}}$; $(R_\lambda, P_\lambda) = (2000, 1550)$ at $t = 2·6\ L_v(0)/u(0)$, which is the time shown.

by comparing results with that of the DIA at $k \to 0$, where the latter is expected to be a good approximation. The choice obtained in this way for $(g_\theta, \tilde{g}_\theta)$ is given in table 1. Figure 5 compares the TFM–EDQNM results utilizing this approach to that given in figure 3. The latter method gives substantially larger values for either theory. The choice of parameters for the velocity field (g or $\tilde{\lambda}$) already matches the Champagne *et al.* (1977) data quite well ($\alpha = 0{\cdot}25$). With regard to the scalar data, other data (namely those of Williams 1974) suggest a somewhat larger constant ($\beta = 0{\cdot}6$) instead of $\beta = 0{\cdot}40$, as found here.

4.2. *Comparison to moderate-Reynolds-number experimental spectra*

Here and in §4.3 we discuss two aspects of moderate-Reynolds-number flows: (i) the comparison of experimental energy-transfer spectra with theory, and (ii) the question of proper equations relating energy and scalar-variance dissipation to an appropriate scalar lengthscale. As noted earlier both TFM and EDQNM yield essentially the same results at large k, but the more comprehensive RNG analysis suggests a TFM-based (really DIA) theory to be more satisfactory at small k. We shall accordingly employ the TFM ($g_v^2 = 1{\cdot}17$, $\tilde{g}_\theta^2 = 0{\cdot}5$, $g_\theta^2 = 1$). However, as observed in §3, a generalization of EDQNM can be made, which more properly treats the small-k range.

The most direct test of closure, such as that described here, would appear to be comparison with wind-tunnel data, which are assured to be reasonably isotropic and homogeneous. Such a comparison meets immediately the problem of the statistics of the initial data. The closure assumes initial Gaussian data, a condition certainly not realized by laboratory experiments. Indeed, any serious consideration of the problem of what initial values of the moments represent conditions as jets merge just exterior to a honeycomb seems a formidable task. As an alternative, one could consider an initial-value problem, in which the energy-transfer spectrum function is given laboratory values, and assume higher-order cumulants to be irrelevant. The validity of a closure then would be judged by how well the evolved spectra matched the experiment decay. Such a procedure has been followed by several authors (Deissler 1979; Cambon *et al.* 1980).

It is difficult to know how sensitive results obtained this way are to the underlying dynamics of the closure. Given $T(k, t = 0)$, it could be that almost any reasonable extrapolation of the initial data for times attainable in the wind tunnel would yield satisfactory results. It is instructive in this regard to consider the calculation by Deissler (1979), who employs an iterative exponential series expansion of the Navier–Stokes equations. His results appear as good in many respects as any obtainable by the much more elaborate closure methods.

In view of these observations, we confine our attention to the following simple problem. Suppose it is known experimentally that $\langle v^2 \rangle \sim t^{-n}$. Then generalizing the arguments invoked by Corrsin (1951) we may relate s and n by a simple use of the energy-conservation equation. The result is $n \approx 2(s+1)/(s+3)$, where at small k, $E(k) \to k^s$ (Lesieur & Schertzer 1978). A reasonable test of a closure would then be to forecast $T(k, t)$ (starting from $T(k, 0) = 0$) and some convenient $E(k, 0)$ (with $E(k, 0) \to k^s$, $k \to 0$). The calculation should start at sufficiently large R_λ so that the experimental value $R_\lambda(t)$ is achieved after transients in $E(k, t)$ have died away. This means that $E(k, t)$ should depend on $E(k, 0)$ only to the extent that both behave like k^s at small k.

Figure 6 shows a comparison of such a calculation with the experiment of Yeh & Van Atta (1973). The experimental data are indicated by the dashed and dot–dashed lines and correspond to two ways of reducing the data. The TFM ($s = 4$) results are

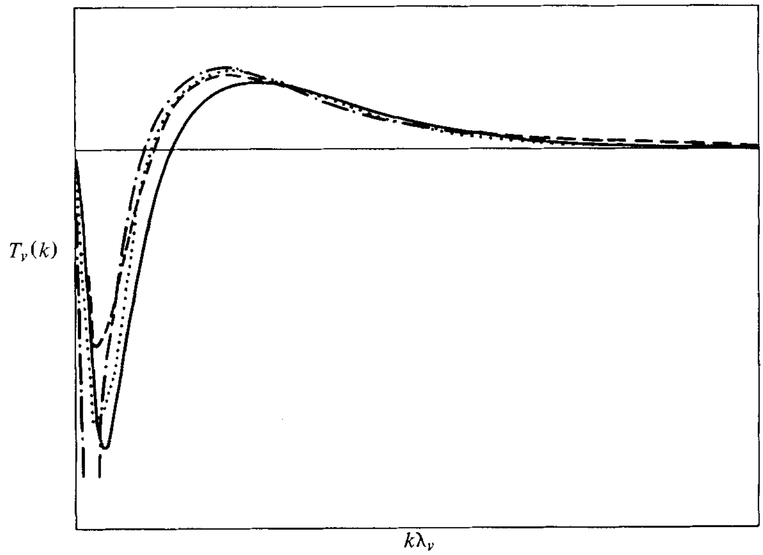


FIGURE 6. Energy-transfer function $T_v(k)$ for TFM at $R_\lambda = 34$. . . . ., $s = 4$ (i.e. $E(k) \to k^4$, $k \to 0$): ——, $s = 1$. – – –, ·–·–·, data of Yeh & Van Atta (1973).

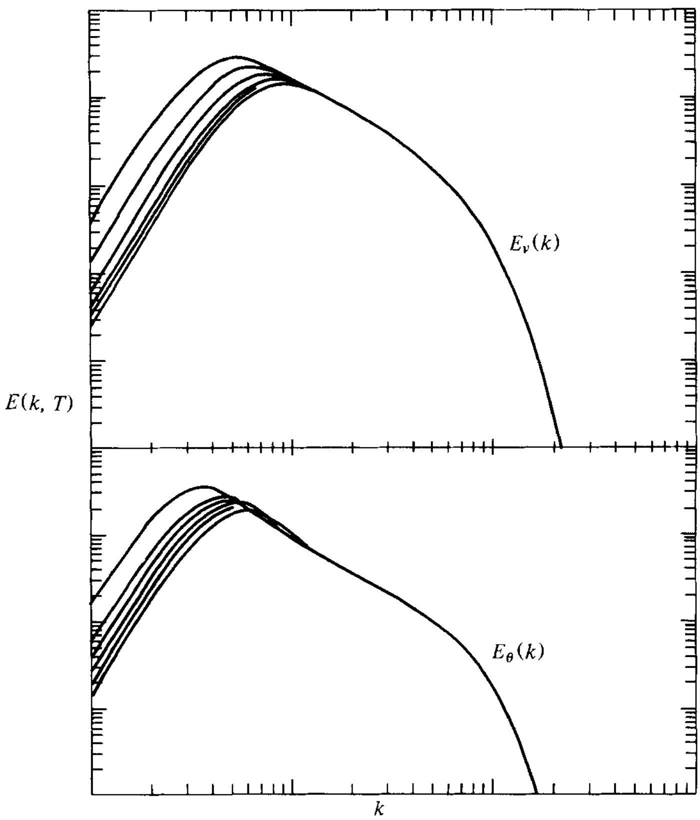


FIGURE 7. Kolmogorov-scaled plots of $E(k)$ and $F(k)$ for $R_\lambda \approx 34$, $P_\lambda \approx 28$ decaying isotropic turbulence, with $s = s' = 4$.

shown as the dotted line. The value $R_\lambda \approx 34$ is obtained for both theory and experiment. Yeh & Van Atta measure $n \approx 1{\cdot}34$, whereas the theoretical calculation (at $R_\lambda \approx 34$) has $n = 1{\cdot}38$. The comparison appears satisfactory. Of perhaps more significance is the sensitivity of these results to the assumed value of s (= 4). The solid line in figure 6 shows the equivalent TFM run with $s = 1$ (again $R_\lambda = 34$). We note the significant shift of $T(k)$ toward larger k (particularly in the energy-containing range). The agreement of the $s = 1$ calculation and the Yeh–Van Atta experiment is clearly less acceptable.

We stress again that the theoretical spectra are self-similar only over large k, $k \gtrsim k_v(t)$, where k_v is the peak-energy wavenumber. Figure 7 gives $E(k, t)$ and $F(k, t)$ with the Kolmogorov scaling. We note an approximately Kolmogorov scaling of these spectra. We have discussed only the velocity field, but a similar discussion applies to the scalar field.

4.3. *Second-order modelling and decay rates at moderate* R_λ

A question of some importance in turbulence modelling is how to specify equations of motion for the dissipation of kinetic energy and scalar variance in terms of quantities used by the models. Such information is necessary to close the second-order moment equations. A widely used procedure is to assume constant values for the normalized decay rates for velocity and scalar fields, defined by

$$\psi = -\frac{2\dot{\epsilon}}{\epsilon}\frac{E(t)}{\epsilon(t)}, \tag{4.2a}$$

$$\psi_\theta = -\frac{2\dot{N}}{N}\frac{\langle\theta'^2\rangle}{N(t)}, \tag{4.2b}$$

and then appeal to experiments, usually free decay, for appropriate values for (ψ, ψ_θ). In this connection, another important useful number is the ratio of velocity and scalar timescales:

$$r(t) = \frac{E/\epsilon}{\langle\theta'^2\rangle/N}. \tag{4.3}$$

If $E \propto t^{-n}$, $\langle\theta'^2\rangle \propto t^{-m}$, $\psi = 2(n+1)/n$, $\psi_\theta = 2(m+1)/m$, $r = m/n$.

Warhaft & Lumley (1978) have called attention to the fact that, whereas those parameters pertaining to the velocity field are fixed with a fair degree of unanimity by free decay experiments, those relating to the scalar are as yet uncertain. They have noted that this variability is probably traceable to the fact that the scalar spectrum's wavenumber centre of gravity (relative to that of $E(k, t)$) varies with experiment. They propose that $r(t)$ should therefore be regarded as a function of $k_v(t)/k_\theta(t)$, where $k_v(t)$ is (as before) the peak k for $E(k, t)$, and k_θ is the same for the scalar field. They were able to verify experimentally this proposition and obtained the functional form $r = f(k_\theta/k_v)$. This was done utilizing a 'mandoline', which injected the scalar field with controlled k_θ downstream from the source of turbulence. A comparison of their data with TFM calculations has also been reported (Larcheveque *et al.* 1980), obtaining reasonable agreement with the experiment.

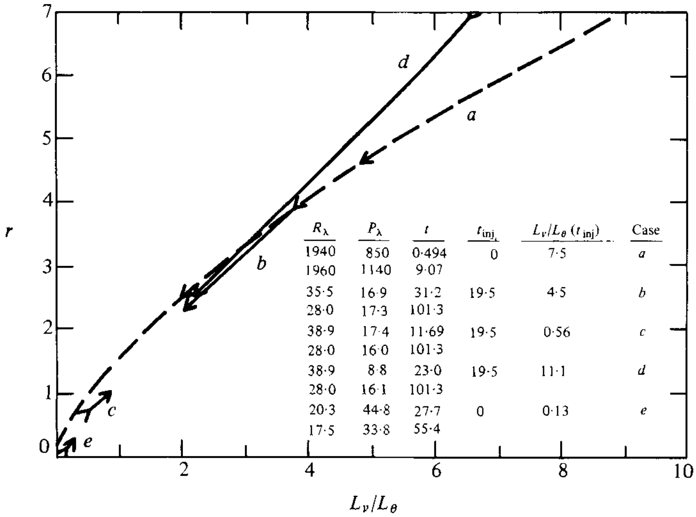
Kerr & Nelkin (1980) have proposed a model based on a relation between the dispersion of a pair of particles and the decay of scalar variance first introduced by Kraichnan (1966) and more recently extended by Larcheveque & Lesieur (1981). Their key assumption is that the scalar variance varies inversely as the cube of the pair (r.m.s.) separation length, which, in turn, should be proportional to a scalar lengthscale $L_s(t)$.

As noted by Kerr & Nelkin (1980), their analysis is expected to be valid only at high Reynolds and Péclet numbers. In addition, an identification of particle-pair separation and a proper scalar lengthscale must be made. It is not yet clear how well their results should be expected to serve as an heuristic explanation of moderate-Reynolds-number experiments such as those of Warhaft & Lumley (1978). Antonopolos-Domis (1981) has more recently reported 'large-eddy simulations' (subgrid-scale numerical simulations) that show that $r(t)$ (see (4.3)) is will represented during decay as linear in the initial microscale ratio λ_v/λ_θ. This characterization of the $r(t)$–lengthscale dependence would appear less useful for our purposes than one utilizing the ratio of integral scales L_v/L_θ, where the Ls are integral scales for (v, θ). This characterization was also suggested by Leslie (private communication 1980). Roughly speaking, we may identify (k_v, k_θ) with $(L_v^{-1}, L_\theta^{-1})$, but the latter really seems a more plausible set of physical variables. Antonopolos-Domis does, in fact, find for his calculations an approximate linear relationship for r versus (k_θ/k_v) (see his figure 13), but this relationship has more scatter than r versus $[\lambda_v/\lambda_\theta$ (initial)].

Here, we extend TFM closure results of Larcheveque *et al.* (1980) to a wider range of $(k_\theta/k_v)_{t=0}$, and discuss in some detail the physics of the scalar decay as this scale parameter varies. In so doing, we bring into focus the relationship of our results to those of Kerr & Nelkin (1980), Sreenivasan *et al.* (1980), and Antonopolos-Domis (1981). Of particular interest is to examine how well the scaling arguments, usually deduced at $R_\lambda \to \infty$, hold up at wind-tunnel R_λ, where most observations are made. Before beginning the numerical discussion, it is worthwhile to inquire under what circumstances unique values of k_θ/k_v and r (defined by (4.3)) exist independent of the initial injection wavenumber $k_\theta(0)$. Following Lesieur & Schertzer (1978), Larcheveque *et al.* (1980) proposed $r = (s'+1)/(s+1)$, where $E(k) \to k^s$, and $F(k) \to k^{s'}$, as $k \to 0$, and $(s, s') < 4$. Underlying the analysis presented there is the assumption that, as $t \to \infty$, $k_\theta/k_v \tilde{\to} 1$. Our numerical calculations indicate that there is indeed a unique $k_\theta/k_v \sim 1$ to which any initial $E(k)$, $F(k)$ tends. This result may seem at first sight somewhat surprising if transfer is entirely local in wavenumber space. For example, if $k_\theta/k_v \gg 1$, and k_θ is in an indefinite inertial range, then $E(kt)$ should, under the assumption of localness, be insensitive to the value of k_v; hence k_θ/k_v would remain arbitrarily large. In reality, the non-local k^4 transfer of (3.2) in this circumstance rapidly transfers F to smaller k where F is initially quite small. The rapidity of this transfer suffices to keep $d(k_\theta/k_v)/dy < 0$ despite the fact that $k_v \sim t^{-p}, p = 2/(s+3)$. It should be noted that the theoretical interpretation of large-m experiments is that $m(t)$ is not in reality constant but slowly decreasing from an experimentally set and arbitrarily large value to a universal value $m \approx 1$.

Figure 8 gives $r(t)$ versus L_v/L_θ for a variety of initial conditions; initial values of k_θ/k_v and R_λ are given in the figure. We include both cases in which θ is injected into pre-established turbulence, as well as cases in which v and θ evolve from Gaussian initial data. The range of R_λ shown is 10–10^3. An examination of the details of the various calculations shows that the functional relationship $r = L_v/L_\theta$ (for $R_\lambda \sim 30$) is relatively insensitive to the initial scalar-injection wavenumber. For example, the point near $L_v/L_\theta = 0{\cdot}2$ represents the decay of a scalar injected at large scales compared with the velocity field $((L_\theta/L_v)\ (t = 0) \approx 7{\cdot}88)$, a dynamically quite different situation from the other cases presented; yet it conforms well to $r = L_v/L_\theta$. These results are consistent with the numerical findings of Antonopolos-Domis (1981) – at least those summarized in his figure 13.

It is of some interest to consider in detail the case in which the initial $k_v/k_\theta \gg 1$. Figure 9 shows $E(k)$ and $F(k)$ for run (e) for which $L_v/L_\theta(t = 0) = 7{\cdot}88$ (other



R_λ	P_λ	t	t_{inj}	$L_v/L_\theta\ (t_{inj})$	Case
1940	850	0·494	0	7·5	*a*
1960	1140	9·07			
35·5	16·9	31·2	19·5	4·5	*b*
28·0	17·3	101·3			
38·9	17·4	11·69	19·5	0·56	*c*
28·0	16·0	101·3			
38·9	8·8	23·0	19·5	11·1	*d*
28·0	16·1	101·3			
20·3	44·8	27·7	0	0·13	*e*
17·5	33·8	55·4			

FIGURE 8. $r(t) = (N/\langle\theta'^2\rangle)/(\epsilon/E)$ versus L_v/L_θ, where Ls denote integral scales for the turbulence and scalar fields. Tracks marked by arrow tails (at earlier) and tips (at later times of evolution) for various runs (a, b, c, d) are shown, as well as the relationship $r = 1{\cdot}63(L_v/L_\theta)^{\frac{2}{3}}$ (– – –). Notice that case (a) (at large R_λ) follows the dashed line over its entire course. Values of parameters R_λ, P_λ, L_v/L_θ at beginning and end of tracks are stated here (s, s') = (4, 4). In addition, t_{inj} is the difference in time (in units of the initial large-scale eddy-circulation time) between Gaussian data for **v** of the injection of Gaussian θ-fluctuations.

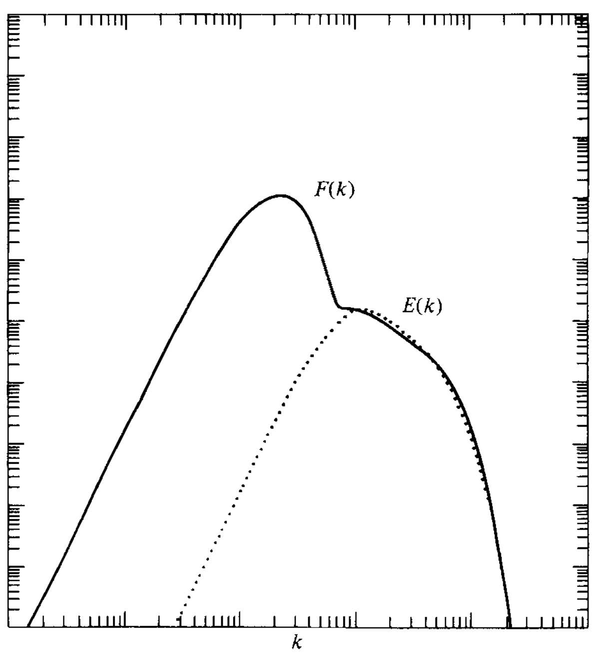


FIGURE 9. $F(k, t = 10)$ (——) and $E(k, t = 8{\cdot}0)$ (. . . . .) versus k for initial conditions (L_θ/L_v) $(t = 0)$ 7·88. Initially $E_v(k) = k^4 \exp(-k)$, $F(k) = k^4 \exp(-10k)$, $R_\lambda \approx 30$. Units are arbitrary.

parameters of this run are given in figure 9). The dynamics of scalar decay for this case appears simply representable as follows. For $k \leqslant k_v(t)$, $F(k)$ decays by eddy diffusivity (the second term in (3.2)); for $k \geqslant k_v$, θ is entrained by v, and the shape of $F(k)$ becomes strongly similar to $E(k)$. As time proceeds, $k_v(t)$ decreases, thereby entraining progressively more θ and mixing it to small scales where it is dissipated. In this case, $F(k, t)$ $(k < k_v)$ stays fixed (except for a slow decay), while $E(k, t)$ etches into it from above. The eddy diffusivity is here well represented by

$$K_{\text{eddy}} = 1{\cdot}16\epsilon_v^{\frac{1}{3}} L_v^{\frac{4}{3}}, \tag{4.4}$$

a value consistent with Larcheveque *et al.* (1980) provided that $Pr^{\mathrm{e}} \approx 0{\cdot}80$.

$F(k)$ for $k \simeq k_v$ appears to approach a power law for about a decade beyond k_v. This feature appears in nearly all our moderate-R_λ calculations (see especially figure 7). Comparison of these runs with those at large R_λ indicates it to be a transitional feature located just a decade interior to the dissipation range. Curiously, in this region, $-d \log F/d \log k \approx \frac{1}{2}(1+\frac{5}{3})$ (i.e., a geometric mean between the inertial-convective and the viscous-convective ranges).

The occurrence at $R_\lambda \approx 30$ and $p_\lambda \approx 30$ of such an 'anomalous inertial range' for $F(k)$ has been noted by Yeh & Van Atta (1973) and by Warhaft & Lumley (1978), the latter suggesting that it may be related to excessively high values of the temperature fluctuation at the heated grid (no such anomaly was found for the mandoline or heated-screen data, which were generally lower in P_λ). Sreenivasan *et al.* (1980), on the other hand, find no such anomalous $F(k)$ range. However, their data, even for the heated grid, had a somewhat lower value of $P_\lambda \approx 20$.

Let us now examine the proposal of Kerr & Nelkin (1980). These authors argue that the behaviour of $\langle \theta'^2 \rangle$ may be deduced from the analysis of the dispersion of a pair of particles, with the r.m.s. separation being proportional to the scalar integral length. Their analysis depends on two key assumptions (i) $\langle \theta'^2 \rangle\ (t) L_\theta^3(t) = \text{constant}$, and (ii) a Richardson's law connecting $\dot{L}_\theta$ and $\epsilon[L_\theta \sim (\epsilon_v/L_\theta)^{\frac{1}{3}}]$. Their equation for $E_\theta(t)$ is

$$\langle \theta'^2 \rangle = [(A+Bt)^{\frac{1}{3}(2-n)}]^{-\frac{9}{2}}, \tag{4.5}$$

where A, B are constants and the decay of $E_v(t) \sim t^{-n}$. First, note that (4.5) is entirely consistent with the scale analysis of EDQNM by Lesieur & Schertzer (1978), provided that the assumption $k_\theta \sim k_v$ is not made and provided that $s' = 2$. Briefly, the analysis proceeds as follows: (i) take $E(k) \sim (k/k_v)^s \epsilon^{\frac{2}{3}} k_v^{-\frac{5}{3}}$ $(k \leqslant k_v)$, $E_v(k) \sim \epsilon_v^{\frac{2}{3}} k^{-\frac{5}{3}}$ $(k \geqslant k_v)$, and $F(k) \sim (k/k_\theta)^{s'} N \epsilon^{-\frac{1}{3}}$ $(k \leqslant k_\theta)$, $F(k) \sim N\epsilon^{-\frac{1}{3}} k^{-\frac{5}{3}}$ $(k \geqslant k_\theta)$; (ii) use (3.1) and (3.2) to infer the invariance of $\epsilon^{\frac{2}{3}} k_v^{-s-\frac{5}{3}}$ and $N\epsilon^{-\frac{1}{3}} k_\theta^{-s'-\frac{5}{3}}$, and then use these constants in the equations $\dot{E} = -\epsilon$ and $\langle \dot{\theta}'^2 \rangle = N$, assuming $(s, s') < 4$. This yields $m = 2(s'+1)/(s+3)$, $n = 2(s+1)/(s+3)$ and $r = (s'+1)/(s+1)$, as already mentioned above, and $L_v \sim t^{2/(s+3)}$, $L_\theta \sim t_0$. Such a calculation clearly assumes large R_λ and P_λ, as well as $Pr \approx 1$. However, as we have noted, the decay power laws derivable from the scale analysis appear to hold down to unexpectedly low R_λ. It is then of some value to ask for the validity of the analysis of Kerr & Nelkin for wind-tunnel R_λ (≈ 30) flows. Figure 10(*b*) gibes log $\langle \theta'^2 \rangle$ versus log $L_\theta(t)$ for two runs in which small-scale Gaussian θ-fields are injected (at $t = 11{\cdot}25$, in units of the large-scale circulation time $\langle n^2 \rangle^{-\frac{1}{2}} L_v$) into decaying turbulence ($E(t) \sim t^{-1{\cdot}38}$). The two cases have $s = 4$ and $s' = (2, 4)$ respectively. These cases are also considered in figure 8 (for $r(t)$ versus L_v/L_θ), where initial $E(k)$ and forms of the injection $F(k, t = 11{\cdot}25)$ are given. At long times $\langle \theta'^2 \rangle\ L_\theta^3$ is constant for $s' = 2$, and $E_\theta L_\theta^{4{\cdot}3}$ for $s' = 4$.

We note the departure from power-law behaviour at short times. Values of

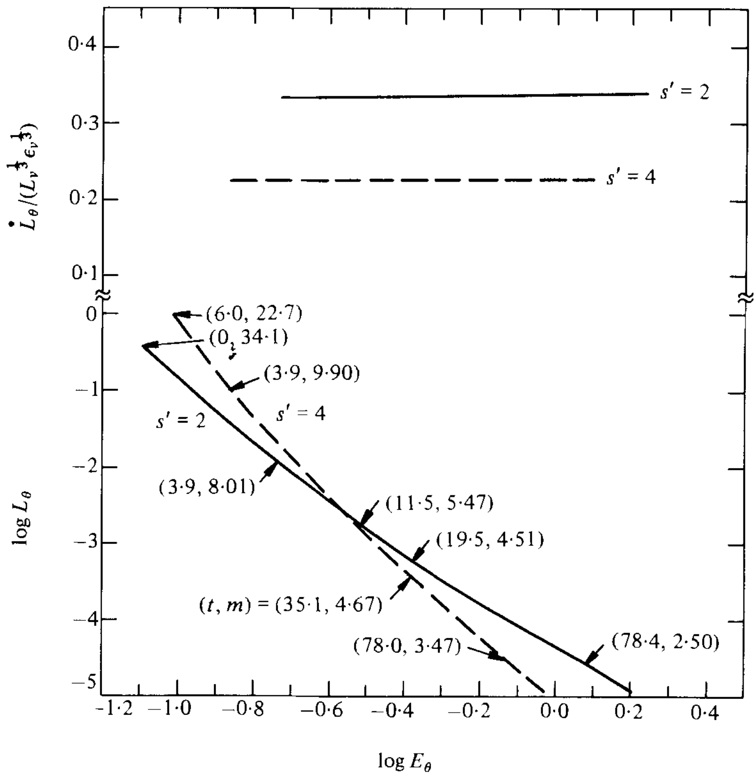


FIGURE 10. $\dot{L}_\theta/\epsilon_v^{\frac{1}{3}} L_\theta^{\frac{1}{3}}$ versus log L_θ (*a*), and log $\langle\theta'^2\rangle$ versus log L_θ (*b*), for two types of initial data: (i) ---, ($E(k) \sim k^4$, $F(k) \sim k^2$, $k \to 0$); (ii) ——, ($E(k) \sim k^4$, $F(k) \sim k^4$, $k \to 0$). In (*b*) the pair of numbers in parentheses is time (in units of initial large-scale circulation time) after injection of the Gaussian θ-field at $t = 11{\cdot}25$ and the current value of m. At $t = 11{\cdot}25$, $F(k) = k^2 \exp[-k(L_v/8{\cdot}81)]$.

$m(t) \equiv t\dot{N}/\langle\theta'^2\rangle$ as well as t are recorded on the figure. Figure 10(*a*) compares $(\dot{L}_\theta/L_\theta)/(\epsilon/L_\theta^2)^{\frac{1}{3}}$ versus log L_θ. We note a confirmation of the proposal of Kerr & Nelkin (1980), and an approximate value of Richardson's constant of 0·35 for $s' = 2$ and 0·27 for $s' = 4$. Calculations at $R_\lambda \approx 10^3$ suggest that these values will increase by about 10 % at large R_λ. It is rather remarkable that the simple scaling analysis should hold at $R_\lambda = 30$; it is, after all, based on inertial-range formulas, and the spectra E_v and E_θ possess only faint suggestions of $k^{-\frac{5}{3}}$ at these small R_λ.

Sreenivasan *et al.* (1980) have reported rather comprehensive experimental results on heated grid and heated screen (= mandoline of Warhaft & Lumley 1978) at moderate $R_\lambda \approx 30$, and $P_\lambda \approx 20$. They have compared their experimental findings with those of Warhaft & Lumley (1978), to other experiments, and to various theoretical models such as those of Corrsin (1964) and Newman, Warhaft & Lumley (1977). Our moderate-R_λ results touch upon their analysis of experimental data, and we briefly comment on this point.

In analysing certain of their data, Sreenivasan *et al.* (1980) use the results of Corrsin (1964):

$$r \sim (L_v/L_\theta)^{\frac{2}{3}}, \tag{4.6}$$

(assuming $Pr \sim 1$). In their analysis of the heated-grid data ($F(k)$ in secular equilibrium with $E(k)$), they use (4.6) to infer a value of $L_v/L_\theta > 1$ (see their figure 6). (Their finding $L_v/L_\theta > 1$ appears at variance with the experiments of Yeh & Van Atta

(1973) and Warhaft & Lumley (1978).) In the TFM this value depends somewhat on (s, s'), but for all our numerical studies so far $L_v/L_\theta < 1$. The analysis underlying their discussion here relating r to L_v/L_θ involves the assumption that both R_λ and P_λ are sufficiently large for appreciable $k^{-\frac{5}{3}}$ ranges for both E_v and E_θ, which is clearly not the case for the experimental data analysed.

As remarked earlier, our moderate-R_λ results are consistent with $r \sim (L_v/L_\theta)^1$ (see also figure 8), in agreement with the suggestion of Warhaft & Lumley (1978) and the numerical simulations of Antonopolos-Domis (1981) (see figure 8). At much larger $R_\lambda = 1{\cdot}5 \times 10^3$, we find curve (*a*) of figure 8, which is well represented by

$$r = (1{\cdot}63 \pm 0{\cdot}001)\,(L_v/L_\theta)^{\frac{2}{3}}. \tag{4.6'}$$

Our results thus indicate that the analyses of Corrsin (1964) and Newman *et al.* (1977), although in agreement with TFM at large R_λ, are not at $R_\lambda \simeq 30$. $r \sim (L_v/L_\theta)^1$ is consistent with an elementary scale analysis of the equations of motion. Thus $\epsilon \sim \langle v^2 \rangle^{\frac{3}{2}}/L_v$, $N \sim \langle \theta^2 \rangle \langle v^2 \rangle^{\frac{1}{2}}/L_\theta$ implies $r \sim L_\theta/L_v$. We should mention that the dependence of r on L_v/L_θ has also been recently investigated by Kolovandin, Luchko & Martynenko (1981), who give a more elaborate argument in favour of $r \sim L_v/L_\theta$. However, if $L_\theta/L_v \ll 1$, only a fraction $(L_\theta/L_v)^{\frac{1}{3}}$ of $\langle v^2 \rangle$ participates in dissipating $\langle \theta'^2 \rangle$. What our present analysis suggests is that at $R_\lambda \sim 30 \sim P_\lambda$, and for L_θ/L_v not too small, the excessive width of $F(k)$ as compared with $E(k)$ is such as to allow all of $\langle v^2 \rangle$ to participate in the dissipation of $F(k)$. The value of the coefficient in (4.6) should depend, at least to a certain extent, on (s, s') $[(s, s') = (4, 4)]$ for (4.6′). Finally, we may compare (4.5′) with the suggested value of Sreenivasan *et al.* (1980), 2.1, and to that proposed by Newman *et al.*, (1977), 1.82.

5. Concluding comments and discussion

The closures described here – TFM and EDQNM – appear to represent in a reasonable manner the dynamics of decaying homogeneous turbulence with scalar fluctuations. An important dynamical ingredient they possess that other simpler procedures frequently omit is the equipartitioning at large scales. This necessarily leads to the correct power laws for decay and suggests that (three-dimensional) spectra commencing like k^4 have a quasi-universal character. The equipartitioning tendency seems vital if one is to model the integral scales correctly. Either method (as we have illustrated) will produce plausible scaling laws of decay rates and integral scales at large Reynolds and Péclet numbers. The closures provide the numbers that the more heuristic approaches cannot. Listed among these for the present application are the modified Richardson constants, the coefficient relating the ratio of decay rates for velocity and scalar fields to the integral scales, and information concerning the eddy Prandtl number. All these quantities depend to a certain degree on the slope of $E(k)$ as $k \to 0$, and the closures quantify the degree of this dependency. The closures further seem to represent faithfully the known spectral features of decay well over a wide range of Pr and for a range of initial conditions.

At the same time, neither procedure may be regarded as very satisfactory from the perspective of theoretical physics. They require scaling factors that must be specified empirically, and have a crude parameterization of the two-time correlation functions (a simple (mis-)use of the fluctuation–dissipation theorem). The errors caused by such a parameterization may be significant in the scalar-dissipation range. Herring & Kerr (1982) have shown that even at quite low R_λ and P_λ the use of the fluctuation–dissipation relation causes a significant inhibition of transfer to large k, especially for the scalar. The DIA is more satisfactory in this respect. The fluctuation–dissipation constraint may be relaxed, but at the expense of increasing the computational labour (as in the strain-based LHDIA: Kraichnan 1977; Herring & Kraichnan 1979), or at the expense of introducing many arbitrary coefficients (relaxing the symmetry of the $D(k, p, q)$).

We have noted that the EDQNM may be regarded as an abridgement of the TFM, and have suggested modifications of the former (see §2.3), which may improve its behaviour at small k. This is important for the computation of quantities such as eddy dissipation that are sensitive to the integral-scale region of the spectrum. Little difference appears at larger k, in the inertial range, and the difference remains small in the dissipation range where the different behaviour in $\mu(k)$ and $\eta(k)$ becomes pronounced (see figure 4). Our conclusions on this point are based on numerical studies here and in Larcheveque *et al.* (1980); there could well be cases where the rapid redistribution of scales needs non-adiabatic features like those in TFM for their accurate description. If such is not the case, it would certainly appear that the EDQNM, being the simpler procedure, would be the proper tool to employ. However, ease of numerical implementation of either procedure is machine-dependent; on a vectorized machine the advantage of the EDQNM in speed of computation may not be so decisive, provided the calculation is within core.

It is a pleasure to record our appreciation to Drs U. Frisch, G. Holloway and R. Kerr for useful and enlightening discussions. We are also grateful to a referee for useful critical comments regarding substance and presentation of material.

The National Center for Atmospheric Research is sponsored by the National Science Foundation.

REFERENCES

André, J. C. & Lesieur, M. 1977 Influence of helicity on the evolution of isotropic turbulence at high Reynolds number. *J. Fluid Mech.* **81**, 187–207.

Antonopolos-Domis, M. A. 1981 Large-eddy simulation of a passive scalar in isotropic turbulence. *J. Fluid Mech.* **104**, 55–79.

Basdevant, C., Lesieur, M. & Sadourny, R. 1978 Subgrid-scale modelling of enstrophy transfer in two-dimensional turbulence. *J. Atmos. Sci.* **35**, 1028–1042.

Batchelor, G. H., Howells, I. D. & Townsend, A. 1959 Small-scale variation of convected quantities like temperature in turbulent fluid. Part 2. The case of large conductivity. *J. Fluid Mech.* **5**, 134–139.

Bell, T. L. & Nelkin, M. 1977 Non linear cascade model for fully developed turbulence. *Phys. Fluids* **20**, 345–350.

Cambon, C., Jeandel, D. & Mathieu, J. 1980 Spectral modelling of homogeneous nonisotropic turbulence. *J. Fluid Mech.* **104**, 247–262.

Champagne, F. H., Friehe, C. A., LaRue, J. C. & Wyngaard, J. C. 1977 Flux measurements, flux estimation techniques and fine-scale turbulence measurements in the unstable surface layer over land. *J. Atmos. Sci.* **34**, 515.

Chollet, J. P. & Lesieur, M. 1981 Parameterization of small scales of three-dimensional isotropic turbulence utilizing spectral closures. *J. Atmos. Sci.* **38**, 2747–2757.

Corrsin, S. 1951 The decay of isotropic temperature fluctuations in an isotropic turbulence. *J. Aero. Sci.* **18**, 417–423.

Corssin, S. 1964 The isotropic turbulent mixer: Part II. Arbitrary Schmidt number. *A.I.Ch.E. J.* **10**, 870–877.

Deissler, R. G. 1979 Decay of homogeneous turbulence from a given state at higher Reynolds number. *Phys. Fluids* **22**, 1852–1856.

Edwards, S. F. 1964 The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239–273.

FORSTER, D., NELSON, D. R. & STEPHEN, M. J. 1977 Large distance and long time properties of a randomly stirred field. *Phys. Rev.* A **16**, 732–749.

FOURNIER, J. D. & FRISCH, U. 1978 *D*-dimensional fully developed turbulence. *Phys. Rev.* A **17**, 747–762.

FRISCH, U., LESIEUR, M. & SCHERTZER, D. 1980 Comment on the quasi-normal Markovian approximation for fully developed turbulence. *J. Fluid Mech.* **97**, 181–192.

FULACHIER, L. & DUMAS, R. 1976 Spectral analogy between temperature and velocity fluctuations in a turbulent boundary layer. *J. Fluid Mech.* **77**, 257–277.

HEISENBERG, W. 1948 Zur statistischen Theorie der Turbulenz. *Z. Phys.* **124**, 622–665.

HERRING, J. R. 1973 Statistical turbulence theory and turbulence phenomenology. In *Proc. Langley Working Conf. on Free Turbulent Shear Flows. NASA* SP321 (1973), Langley Research Center, Langley, VA (available from NTIS as N73-2815415GA).

HERRING, J. R. 1974 Approach of an axisymmetric turbulence to isotropy. *Phys. Fluids* **17**, 859–872.

HERRING, J. R. & KERR, R. M. 1982 Comparison of direct numerical simulation with predictions of two-point closures for isotropic turbulence convecting a passive scalar. *J. Fluid Mech.* **118**, 205–219.

HERRING, J. R. & KRAICHNAN, R. H. 1972 Comparison of some approximations for isotropic turbulence. In *Statistical Models and Turbulence* (ed. M. Rosenblatt & C. Van Atta). *Lecture Notes in Physics*, vol. 12, pp. 148–194. Springer.

HERRING, J. R. & KRAICHNAN, R. H. 1979 Numerical comparison of velocity-based and strain-based Lagrangian-history turbulence approximations. *J. Fluid Mech.* **91**, 381–397.

HILL, R. J. 1978 Models of the scalar spectrum for turbulent advection. *J. Fluid Mech.* **88**, 541–562.

HOWELLS, I. D. 1960 An approximate equation for the spectrum of a conserved scalar quantity in a turbulent fluid. *J. Fluid Mech.* **9**, 104–106.

KERR, R. M. & NELKIN, M. 1980 The decay of scalar variance simply expressed in terms of a modified Richardson law for particle dispersion. Preprint, submitted to *Phys, Fluids.*

KOLOVANDIN, B. A., LUCHKO, N. N. & MARTYNENKO, O. G. 1981 Modeling of homogeneous turbulent scalar field dynamics. In *Proc. 3rd Symp. on Turbulent Shear Flow, Sept.* 9–11, *University of California, Davis, CA.*, pp. 15.7–15.12.

KRAICHNAN, R. H. 1958 Irreversible statistical mechanics of incompressible hydrodynamic turbulence. *Phys. Rev.* **109**, 1407–1422.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497–543.

KRAICHNAN, R. H. 1965 Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8**, 575–598 (erratum **9**, 1884).

KRAICHNAN, R. H. 1966 Dispersion of particle pairs in homogeneous turbulence. *Phys. Fluids* **9**, 1937–1943.

KRAICHNAN, R. H. 1968 Small scale structure convected by turbulence. *Phys. Fluids* **11**, 945–953.

KRAICHNAN, R. H. 1971 An almost-Markovian Galilean-invariant turbulence model. *J. Fluid Mech.* **47**, 513–524.

KRAICHNAN, R. H. 1976 Eddy viscosity in two and three dimensions. *J. Atmos. Sci.* **33**, 1521–1536.

KRAICHNAN, R. H. 1977 Eulerian and Lagrangian renormalization in turbulence theory. *J. Fluid Mech.* **83**, 349–374.

KRAICHNAN, R. H. & HERRING, J. R. 1978 A strain-based Lagrangian-history turbulence theory. *J. Fluid Mech.* **88**, 355–367.

LARCHEVEQUE, M., CHOLLET, J. P., HERRING, J. R., LESIEUR, M., NEWMAN, G. R. & SCHERTZER, D. 1980 Two-point closure applied to a passive scalar in decaying isotropic turbulence. In *Turbulent Shear Flows*, 2 (ed. L. J. S. Bradbury, F. Durst, B. E. Launder, F. W. Schmidt & J. H. Whitelaw), pp. 50–65. Springer.

LARCHEVEQUE, M. & LESIEUR, M. 1981 The application of eddy-damped Markovian closures to the problem of dispersion of particle pairs. *J. Méc.* **20**, 113–134.

LEE, T. D. 1952 On some statistical properties of hydrodynamical and magneto-hydrodynamical fields. *Q. Appl. Math.* **10**, 69–74.

LEITH, C. E. 1968 Diffusion approximation for turbulent scalar fields. *Phys. Fluids* **11**, 1612–1617.

LESIEUR, M. & CHOLLET, J.-P. 1980 In *Bifurcation and Nonlinear Eigenvalue Problems* (ed. C. Bardos, M. Schatzmann & J. M. Lasry). *Lecture Notes in Mathematics*, vol. 782, pp. 101–121. Springer.

LESIEUR, M. & SCHERTZER, D. 1978 Amortissement autosimilaire d'une turbulence à grand nombre de Reynolds. *J. Méc.* **17**, 607–646.

LESIEUR, M., SOMMERIA, J. & HOLLOWAY, G. 1981 Zones inertielles du spectre d'un contaminant passif en turbulence bidimensionnelle. *C. R. Acad. Sci. Paris, Sér.* II **292**, 271–274.

LESLIE, D. C. 1973 *Developments in the Theory of Turbulence.* Clarendon.

LESLIE, D. & QUARINI, G. L. 1978 The application of turbulence theory to the formulation of subgrid modelling procedures. *J. Fluid Mech.* **91**, 65–91.

MESTAYER, P. 1980 De la structure fine des champs turbulents dynamique et thermique pleinement développés en couche limite. Thèse d'Etat, Université d'Aix-Marseille II.

MONIN, A. S. & YAGLOM, A. M. 1975 *Statistical Fluid Mechanics*, vol. 2. M.I.T. Press.

NEWMAN, G. R. & HERRING, J. R. 1979 A test field model study of a passive scalar in isotropic turbulence. *J. Fluid Mech.* **94**, 163–194.

NEWMAN, G. R., WARHAFT, Z. & LUMLEY, J. L. 1977 The decay of temperature fluctuations in isotropic turbulence. In *Proc.* 6*th Australian Hydraulics and Fluid Mechanics Conference, Adelaide.*

O'BRIEN, E. F. & FRANCIS, G. C. 1962 A consequence of the zeroth fourth cumulant approximation. *J. Fluid Mech.* **13**, 369–382.

OBOUKHOV, A. M. 1941 On the distribution of energy in the spectrum of a turbulent flow. *Dokl. Akad. Nauk SSSR* **32**, 22–24.

OGURA, Y. 1962 The dependency of eddy diffusivity on the fluid Prandtl number. *Adv. Geophys.* **6**, 175–177.

ORSZAG, S. A. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363–386.

ORSZAG, S. A. 1974 Statistical theory of turbulence. In *Proc.* 1973 *Les Houches Summer School* (ed. R. Balian & J.-L. Peabe), pp. 237–374. Gordon & Breach.

PHYTHIAN, R. 1969 Self-consistent perturbation series for stationary homogeneous turbulence. *J. Phys.* A: *Gen. Phys.* **2**, 181–192.

POUQUET, A., LESIEUR, M. & ANDRÉ, J. C. 1975 High Reynolds number simulation of two-dimensional turbulence using a stochastic model. *J. Fluid Mech.* **72**, 305–319.

QUARINI, G. L. 1976 Evaluation of inertial-convective range scalar transfer by re-appraisal of existing closures. Preprint.

ROSE, H. & SULEM, P. L. 1978 Fully developed turbulence and statistical mechanics. *J. Phys.* (*Paris*) **39**, 441–483.

ROTTA, J. 1951 Statistische Theorie nichthomogener Turbulenz. *Z. Phys.* **129**, 547–572.

SCHERTZER, D. 1980 Comportements auto-similaires en turbulence homogène isotrope. *C.R. Acad. Sci. Paris* **280**, 277.

SCHUMANN, U. & HERRING, J. R. 1976 Axisymmetric homogeneous turbulence: a comparison of direct spectral simulations with the direct-interaction approximation. *J. Fluid Mech.* **76**, 755–782.

SREENIVASAN, K. R., TAVOULARIS, S., HENRY, R. & CORRSIN, S. 1980 Temperature fluctuations and scales in grid-generated turbulence. *J. Fluid Mech.* **100**, 597–621.

SULEM, P. L., LESIEUR, M. & FRISCH, U. 1975 Le 'Test Field Model' interprété comme méthode de fermeture des équations de la turbulence. *Ann. Geophys.* **31**, 487–495.

TATSUMI, T., KIDA, S. & MIZUSHIMA, J. 1978 The multiple-scale cumulant expansion for isotropic turbulence. *J. Fluid Mech.* **85**, 97–142.

WARHAFT, Z. 1980 An experimental study of the effect of uniform strain on thermal fluctuations in grid-generated turbulence. *J. Fluid Mech.* **99**, 545–573.

WARHAFT, Z. & LUMLEY, J. L. 1978 An experimental study of the decay of temperature fluctuations in grid-generated turbulence. *J. Fluid Mech.* **88**, 659–684.

WILLIAMS, R. M. 1974 High frequency temperature and velocity fluctuations in the atmospheric boundary layer. Ph.D. thesis, Oregon State University.

YEH, T. T. & VAN ATTA, C. W. 1973 Spectral transfer of scalar and velocity fields in heated-grid turbulence. *J. Fluid Mech.* **58**, 233–261.